PROJECT 5

THE COLLAPSE OF GALLOPING GERTIE

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FIGURE 1 The Tacoma Narrows Bridge—before and after the collapse

You've probably seen the movies. The bridge begins to wobble from side to side. The oscillations get larger and larger. Leonard Coatsworth leaves his car with his dog Tubby inside and crawls on his hand and knees off the bridge to safety. Suddenly, the bridge collapses. The date is November 7, 1940, only four months after its grand opening. The collapse came as no surprise because the Tacoma Narrows Bridge,* or "Galloping Gertie," as it was fondly called by local residents, was notorious—even before it opened for traffic—for a swaying and vertical undulating motion of its roadway caused by the peculiar wind currents that pass through the narrows. See Figure 1.

After working through the problems in Chapter 5, you might suspect that resonance was the culprit. Somehow the forcing of the wind and the natural frequency of the suspension cables coincide, and thus the amplitude of the forced system (in this case the bridge) grew without bound, eventually causing the bridge to fall. But recent work by Lazer and McKenna suggests that the phenomenon that caused the bridge to fail was more complex than resonance.† Most of the models presented in their work are beyond the scope of this project. The **Still Curious?** section at the end of this project provides an introduction to one of their more elementary models.

In the previous edition of this text, Gilbert Lewis proposed a new model based on the work of Lazer and McKenna. This model illustrates one nonlinear mechanism that could have led to the demise of Galloping Gertie.

Imagine one cable of the suspension bridge hanging vertically. In some ways the cable is much like a very stiff spring. It has a natural length, the end of which we will place at $x = 0$. When a cable is stretched beyond this length ($x > 0$), it exerts a force in the upward direction; when compressed ($x < 0$), it exerts a force in the downward direction. But a cable is not a spring, and it is relatively easy to convince yourself that the upward restitution force that results from stretching the cable is greater than the downward force that comes from compressing it. In other words, we want to model the cable like a spring using Hooke's law with different spring constants depending on whether $x < 0$ or $x > 0$. Let a denote the

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*The original Tacoma Narrows Bridge connected the city of Tacoma, Washington, and Gig Harbor, Washington.

†A.C. Lazer and P.J. McKenna, Large Amplitude Periodic Oscillations in Suspension Bridges: Some New Connections with Nonlinear Analysis, SIAM Review 32 (December 1990): 537–578.

spring constant for compression and b the spring constant for stretching so that $0 < a < b$. Define

$$F(x) = \begin{cases} ax, & x < 0 \\ bx, & x \geq 0. \end{cases} \tag{1}$$

In the absence of damping, our differential equation then becomes

$$mx'' + F(x) = g(t), \tag{2}$$

where the function $g(t)$ represents the forcing due to the wind.

PROBLEM 1 (CD). To illustrate the core issue underlying this phenomenon, suppose in (2) that $m = 1$, $a = 1$, $b = 4$, $g(t) = \sin 4t$, and initial conditions $x(0) = 0$, $x'(0) = \alpha$. Use the **Tacoma Bridge Tool** on the *DE Tools* CD to plot solutions of (2) on the interval $0 \leq t \leq 100$ for a variety of values of α. What do you observe and what might these observations say about the fate of the bridge?

As was mentioned above, the model given in (2) is nonlinear. But the nonlinearity arises because $F(x)$ is piecewise linear. Thus we can find partial solutions that are defined over time intervals where the solution $x(t)$ does not change sign.

PROBLEM 2. Use the parameters and initial conditions given in Problem 1 but assume that $\alpha > 0$. If $F(x) = 4x$ for t small, show that

$$x(t) = \frac{1}{6} \sin 2t \, [3\alpha + 1 - \cos 2t]$$

is a solution of (2) on the interval $0 \leq t \leq \pi/2$. Note that in addition to showing that $x(t)$ satisfies the differential equation and initial conditions you must also show that $x(\pi/2) = 0$. Plot this function on the given time interval. Find $x'(\pi/2)$ and show that it is negative.

For the next time interval we use $F(x) = x$.

PROBLEM 3. Use the values of $x(\pi/2)$ and $x'(\pi/2)$ obtained in Problem 2 and show that

$$x(t) = \cos t \left[\left(\alpha + \frac{2}{5} \right) - \frac{4}{15} \sin t \cos 2t \right]$$

is a solution of (2) on the interval $\pi/2 \leq t \leq 3\pi/2$.

We refer to each solution piece as a **cycle.** Thus each cycle corresponds to either a positive or a negative displacement of the bridge.

PROBLEM 4 (CAS). Use your solutions in Problems 2 and 3 to plot $x(t)$ on the interval $0 \leq t \leq 3\pi/2$. How does the velocity at $t = 3\pi/2$ compare with the velocity at $t = 0$?

Clearly, we could continue in this manner, successively solving the appropriate linear differential equation using the initial conditions $x = 0$ and a new value

x' calculated from the final x' of the previous solution. A careful analysis of this reveals that the amplitude of each successive cycle increases by a constant rate proportional to $\frac{2}{15}$. Thus as $t \to \infty$, the amplitudes of the oscillations grow without bound. This is what you should have observed in Problem 1.

It is instructive to see what happens for some other values of a and b in (1).

PROBLEM 5 (CD). Consider the three cases of (1) where $b = 1$, $a = 4$; $b = 64$, $a = 4$; and $b = 36$, $a = 25$. Notice that in the first case the condition $0 < a < b$ is not satisfied. Again using $g(t) = \sin 4t$, $m = 1$, and initial conditions $x(0) = 0$, $x'(0) = 1$, plot the solution of (2) on the interval $0 \le t \le 100$ in each of the three cases. Describe the long-term behavior of $x(t)$ in each of the three cases.

STILL CURIOUS?

The paper by Lazer and McKenna referred to above presents a somewhat different model. In this section we briefly discuss that model. Consider the differential equation

$$x'' + \beta x' + F(x) = -g + \lambda \sin \omega t \tag{3}$$

where β is the damping constant, g is the acceleration due to gravity, and the function F is defined in (1). The new features in this model are the damping term $\beta x'$ and the forcing term $\lambda \sin \omega t$ due to the wind. In the next two problems we take $a = 17$, $b = 13$, $\beta = 0.01$, and $g = 10$.

PROBLEM 6 (CD). Let's begin by seeing what happens when $\lambda = 0$. This represents the state of the bridge when the wind is not blowing. Plot the solution of (3) with initial conditions $x(0) = x_0 < 0$, $x'(0) = 0$ on the interval $0 \le t \le 50$ and describe what happens to the bridge.

PROBLEM 7 (CD). Now let's see what might happen when the wind blows. Let $\omega = 4$ and $\lambda = 0.04$. Plot solutions of (3) with initial conditions $x(0) = -\frac{10}{17}$, $x'(0) = 0$ on the interval $0 \le t \le 50$ and describe what happens to the bridge. Does $x(t)$ ever cross the t-axis? What part of the piecewise defined function $F(x)$ is relevant in this case? What are the physical implications of this property?

In Problem 7 you probably noticed that $x(t) < 0$ for all t. Thus only one "piece" of $F(x)$ is relevant. In other words, we are really solving the differential equation

$$x'' + 0.01x' + 17x = -10 + 0.04 \sin 4t. \tag{4}$$

PROBLEM 8 (CAS). Solve the differential equation in (4) with the initial conditions $x(0) = x_0 < 0$, $x'(0) = 0$. Indicate the transient and steady-state terms as $t \to \infty$.

In Problem 7 the small value of λ means that the wind is not blowing very hard. Let's see what happens as this parameter grows and the wind begins to howl.

PROBLEM 9 (CD). Let $\lambda = 0.2$ and $x'(0) = 0$. For each of the initial displacements $x(0) = -0.5, -0.4, \ldots, 0.4, 0.5$, plot solution curves on the interval $0 \leq t \leq 50$. Describe what happens to the oscillations as the values of $x(0)$ increase. Interpret this in terms of the bridge system being modeled.

THE REST OF THE STORY

The bridge over the Tacoma Narrows was eventually rebuilt. It opened in October 1950, 10 years after the collapse. Notice in Figure 2 that the original bridge, whose fatal flaw was its light and graceful design, was replaced by one whose span was greatly stiffened by truss work. But the "new" 1950 bridge is outdated by current highway standards; a new suspension bridge is currently under construction parallel to the existing bridge. See Figure 3.

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FIGURE 2 Tacoma Narrows Bridge rebuilt in 1950

Photo courtesy of Tacoma Narrows Constructors

FIGURE 3 Design of parallel bridges; the addition is scheduled to open in 2007