Photo by Nick Wilson

Dai Fujiwara and Bill Thurston on the runway at Paris Fashion Week, 2010.

High Fashion Meets Higher Mathematics

Kelly Delp

This author is but one of many who was influenced by Bill Thurston, and saddened by his death on August 21, 2012. She is grateful for the time they spent playing with mathematics.

Try the following experiment. Get a tangerine and attempt to take the peel off in one piece. Lay the peel flat and see what you notice about the shape. Repeat several times. This can be done with many types of citrus fruit. Clementines work especially well.

Cornell mathematics professor William P. Thurston used this experiment to help students understand the geometry of surfaces. Thurston, who won the Fields Medal in 1982, was well known for his geometric insight. In the early 1980s he made a conjecture, called the *geometrization conjecture*, about the possible geometries for three-dimensional manifolds. Informally, an n-dimensional manifold is a space that locally looks like $\mathbb{R}^n$.

Photo by William Thurston

Image from Peeling the Oranges Reception, Paris Fashion Week, March 2010.

Although Thurston proved the conjecture for large classes of three-manifolds, the general case remained one of the most important outstanding problems in geometry and topology for 20 years. In 2003 Grigori Perelman proved the conjecture. The geometrization conjecture implies the Poincaré conjecture, so with his solution Perelman became the first to solve one of the famed Clay Millennium Problems. (The November 2009 issue of *Math Horizons* ran a feature on Perelman.)

The story of Thurston's geometrization conjecture and the resolution of the Poincaré conjecture drew attention from reporters and other writers outside of the mathematical community. One person who happened upon an account of Thurston and his work was the creative director of House of Issey Miyake, fashion designer Dai Fujiwara. In a letter to Thurston, Fujiwara described how he felt a connection with the geometer, as he had used the same technique of peeling fruit to explain clothing design to students new to the subject. Designers also practice the art of shaping surfaces from two-dimensional pieces.

Fujiwara felt Thurston's three-dimensional geometries could provide a theme for Issey Miyake's ready-to-wear fashion line. Thurston, who in 1991 had organized (along with his mother, Margaret Thurston) what was perhaps the first mathematical sewing class as part of the Geometry and Imagination Workshop, agreed there was potential for connection. Thus the collaboration was born. The Issey Miyake collection inspired by Thurston's eight geometries debuted on the runway at Paris Fashion Week in spring 2010.

Geometry of Surfaces

Before discussing the fashion show and the geometry of three-manifolds, we will discuss geometries of two-dimensional objects, or *surfaces*. Examples of surfaces include the sphere, the torus, and the Möbius band. Any (orientable) surface can be embedded in $\mathbb{R}^3$, and this allows us to measure distances on the surface. Let's think about the sphere with radius one

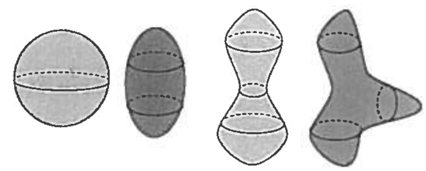


Figure 1. Spheres.

centered at the origin. This sphere is described by the familiar equation $x^2+y^2+z^2=1$. Choose any two points on the sphere, say the north pole $(0,0,1)$, and the south pole $(0,0,-1)$.

There are two natural ways to define the distance between these points. The first way is to assign the distance between points to be the usual Euclidean distance in $\mathbb{R}^3$. In this metric, the distance between the poles is two, the length of the diameter.

For another metric, assign the distance to be the minimum length of any path on the sphere that starts at one pole and ends at the other; this would be the length of the shortest arc of a great circle

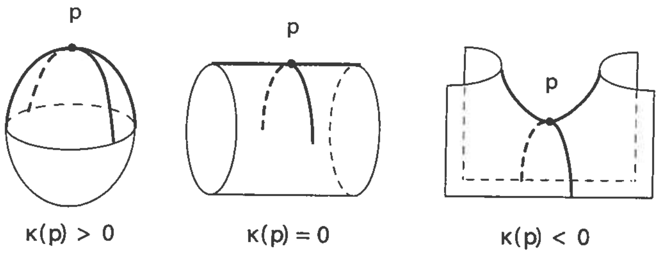
Beauty is truth, truth beauty—that is all ye know on earth, and all ye need to know.

This famous and provocative quotation of John Keats is echoed on the emblem of the Institute for Advanced Study, where I took my first job after graduate school. After reading an account of my mathematical discoveries concerning eight geometries that shape all three-dimensional topology, Dai Fujiwara made the leap to write to me, saying that he felt in his bones that my insights could give inspiration to his design team at Issey Mayake. He observed that we are both trying to understand the best three-dimensional forms of two-dimensional surfaces, and he noted that we each, independently, had come around to asking our students to peel oranges to explore these relationships. This resonated strongly with me, for I have long been fascinated (from a distance) by the art of clothing design and its connections to mathematics.

Many people think of mathematics as austere and self-contained. To the contrary, mathematics is a very rich and very human subject, an art that enables us to see and understand deep interconnections in the world. The best mathematics uses the whole mind, embraces human sensibility, and is not at all limited to the small portion of our brains that calculates and manipulates symbols. Through pursuing beauty we find truth, and where we find truth we discover incredible beauty.

The roots of creativity tap deep within to a place we all share, and I was thrilled that Dai Fujiwara recognized the deep commonality underlying his efforts and mine. Despite literally and figuratively training and working on opposite ends of the earth, we had a wonderful exchange of ideas when he visited me at Cornell. I feel both humbled and honored that he has taken up the challenge to create beautiful clothing inspired by the beautiful theory that is dear to my heart.

William P. Thurston



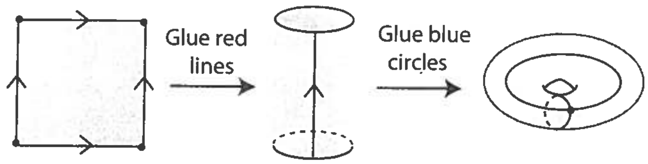


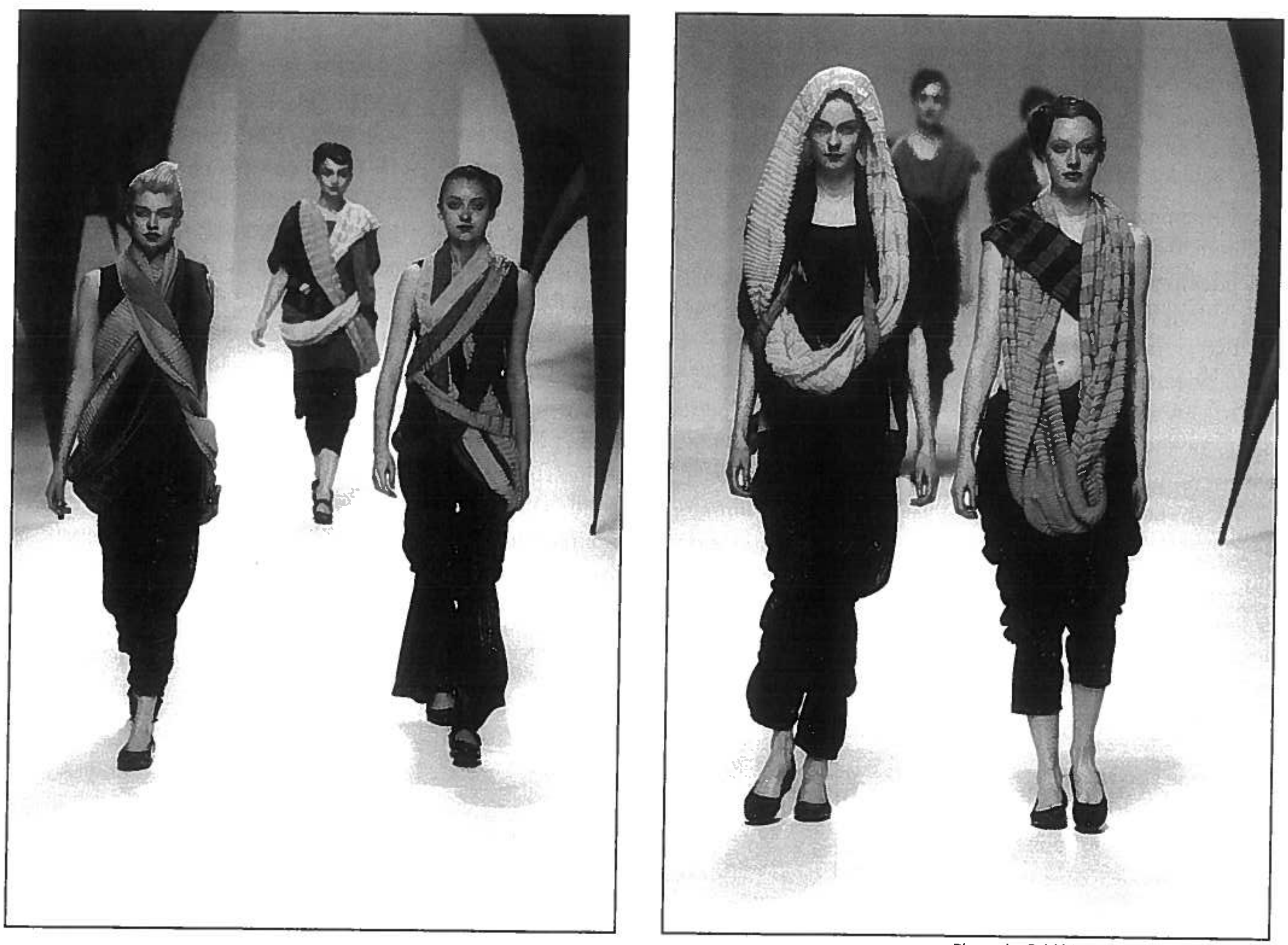
Figure 2. Gaussian curvature, left.
Figure 3. Building a torus, above.

between them. Now the distance between the poles would be π. This latter metric is more appropriate; after all, when traveling from Buffalo to Sydney, the best way is not by drilling through the center of the earth.

Two manifolds are *topologically equivalent* if there is a continuous bijection, with continuous inverse, between them. The bijection is called a *homeomorphism*, and we say the manifolds are *homeomorphic*. Under this equivalence relation, all of the surfaces in figure 1 are spheres.

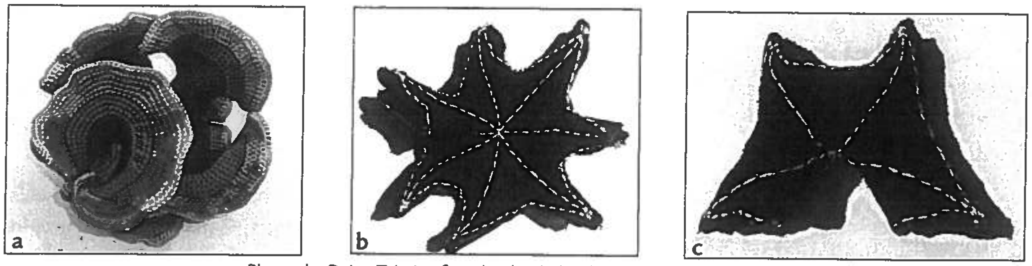
Each of these spheres can be equipped with a metric from $\mathbb{R}^3$, as previously described, by measuring the shortest path in the surface. Even though they are topologically equivalent, as metric spaces they are very different. Two surfaces are *metrically equivalent* if there is a distance-preserving map, called an *isometry*, between them.

One quantity that is preserved under isometries is Gaussian curvature. Recall that the Gaussian curvature is a function k from a surface S to the real numbers, where $k(p)$ is the product of the principal curvatures at p; roughly speaking, $k(p)$ gives a measure of the amount and type of bending of the surface at a point p. At a point of positive curvature, all of the (locally length minimizing) curves through p bend in the same direction; in



Photos by Frédérique Dumoulin/Issey Miyake

Issey Miyake's spring 2010 collection on the runway at Paris Fashion Week.



Photos by Daina Taimina from her book *Crocheting Adventures with Hyperbolic Planes* (AK Peters, 2009)

Figure 4. Daina Taimina's hyperbolic crocheting: from left, hyperbolic plane, hyperbolic octagon, hyperbolic pair of pants.

zero curvature there is a straight line in the surface through p; and in negative curvature, the surface has curves that bend in opposite directions. (See figure 2.)

You should be able to identify points of negative curvature in both the green and yellow spheres in figure 1. The purple sphere has only positive curvature, though the curvature is greater at the north pole than at the equator. The blue sphere is the most symmetric and has constant curvature $k(p) = 1$ for all p.

The round sphere is one of three model two-dimensional geometries. The other two model geometries are the Euclidean plane, which has constant zero curvature, and the hyperbolic plane, which has constant curvature -1. Every (compact and smooth) surface supports a metric of exactly one type of constant curvature: positive, negative, or zero. Although the sphere has many different metrics, it cannot have a Euclidean or hyperbolic metric. We give examples of surfaces of the latter two types.

Euclidean surfaces, such as a torus, can be constructed from pieces of the Euclidean plane. Start with a rectangle in the Euclidean plane. A sheet of paper works nicely as a model. Tape together opposite sides of the piece of paper (mathematically, this is done by creating a quotient space). If we stop at this point, we have a Euclidean cylinder. If we identify the opposite two boundary circles, we will create a torus with a Euclidean metric inherited from the original rectangle.

Of course, if you try to do this with your paper cylinder, you will find it impossible. You can come close by folding and creasing the cylinder, but the final object does not look very much like a torus. We do not allow creasing as a legal construction technique, as the corresponding mathematical object would not have a tangent plane along the crease. If we had a fourth dimension to bend into, we could tape together the opposite circles without distorting the metric.

Finally, we will describe an example of a hyperbolic surface. Figure 4a pictures a crocheted model of a piece of the hyperbolic plane, conceived and constructed by Daina Taimina. We point out one difference between the hyperbolic plane and the Euclidean plane, which can be seen in this model. If one crocheted a Euclidean disc, the number of stitches in concentric circles would increase linearly; this is because the circumference of a circle in the Euclidean plane is a linear function of the radius—$2\pi r$. In the hyperbolic plane, the circumference of the circle grows *exponentially* with the radius—$2\pi \sinh(r)$—creating the wavy surface seen in figure 4a.

Figure 4b shows a regular octagon with 45-degree interior angles in the hyperbolic plane. Note that such a polygon could not occur in the Euclidean plane, where regular octagons have interior angles of 135 degrees. If we identify every other side (the ones marked with black Velcro in figure 4b) of this octagon, we create a *hyperbolic pair of pants*, shown in figure 4c. Note that "pair of pants" is the name geometric topologists use to refer to surfaces of this homeomorphism type, which are important building blocks for all hyperbolic surfaces.

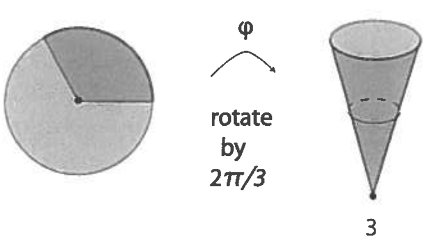


Figure 5. Euclidean cone.

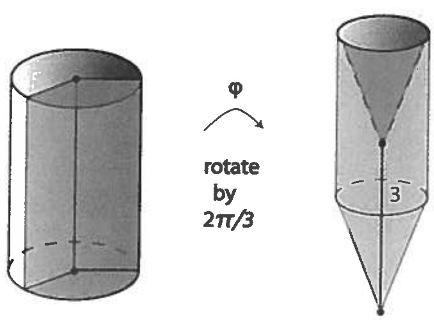
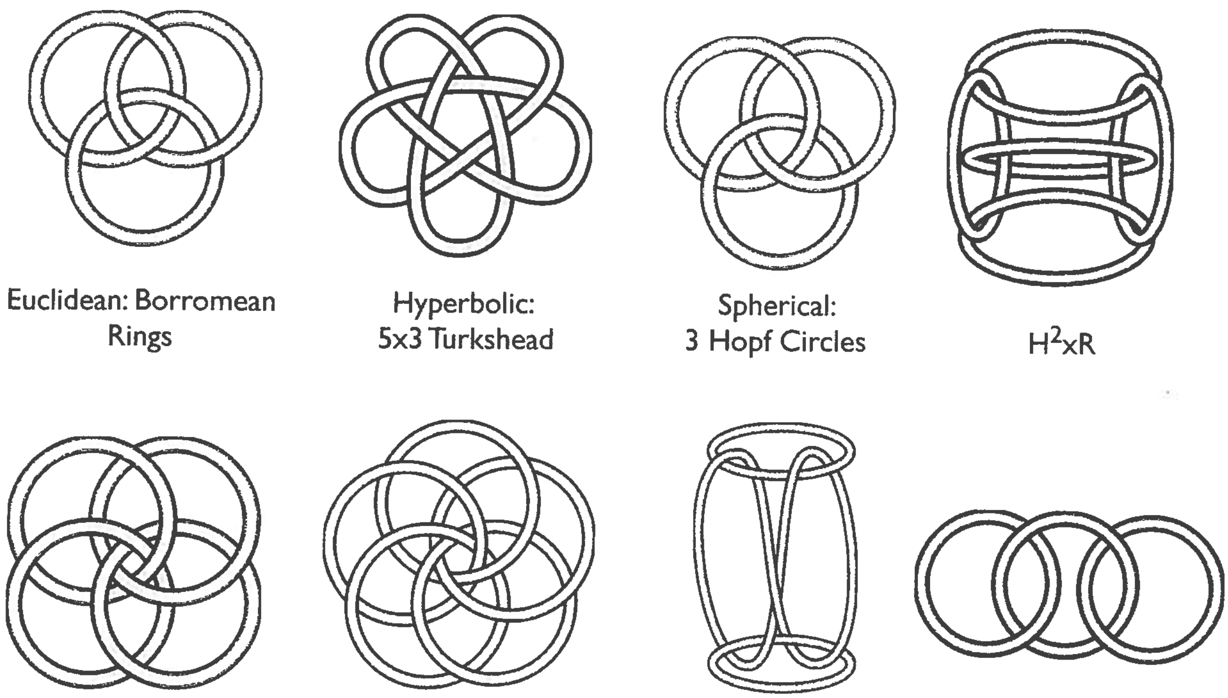


Figure 6. Three-dimensional Euclidean orbifold.

Orbifolds, Three-Dimensional Geometries, and Design

Essentially, Thurston's geometrization conjecture states that any three-manifold can be decomposed into finitely many pieces, each of which supports a metric modeled on one of eight geometries: the three-dimensional analogue of spherical, hyperbolic, or Euclidean space, or one of five other possible geometries:



Euclidean: Borromean Rings
Hyperbolic: 5x3 Turkshead
Spherical: 3 Hopf Circles
H^2xR
Nil: 4 Hopf Circles
SL(2,R): 5 Hopf Circles
Solv
S^2xR

Figure 7. Illustrations of orbifold representatives of the eight geometries.

$S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, *Nil*, *Solv*, or *universal cover of* $SL(2,\mathbb{R})$. If a closed manifold supports one of these eight geometries, it cannot support a metric of any of the other seven types.

An example of a Euclidean three-manifold is the three-dimensional torus. To mathematically construct this manifold, start with a solid Euclidean cube, which can be described as the set of points (x,y,z) in $\mathbb{R}^3$ such that $0 \le x,y,z \le 1$. Identify opposite faces of the cube by distance-preserving maps. More explicitly, identify the top face $(z=1)$ to the bottom face $(z=0)$, by defining an equivalence relation on the cube that identifies $(x,y,0)$ with $(x,y,1)$. Analogous identifications can be made between the front and back faces, and with the left and right faces.

The room you are sitting in, if it is roughly cubical, provides a good model for this space. Once the floor and the ceiling have been identified, when you look straight up you will see the bottom of your feet. If you would like to experience different geometries of three-dimensional manifolds, try Jeff Week's program *Curved Spaces,* available at his website: *http://geometrygames.org/.*

Using the eight three-dimensional geometries as inspiration for a fashion line seemed like a difficult endeavor; clothing is essentially two-dimensional. To learn about these geometries, and to exchange ideas, Fujiwara visited Thurston at Cornell. After returning to Japan, he continued to exchange ideas with Thurston, and the topologist Kazushi Ahara from Meiji University gave a series of lectures about the geometries to the design team. The designers were a somewhat apprehensive audience, and Ahara promised not to use certain words, such as "equation" or "trigonometric function" during his lectures.

So what was the result of the collaboration? How did Fujiwara accomplish the difficult task of representing the eight geometries in the Issey Mayake line of fashion? At the time of writing this article, several videos from the fashion show can be found on YouTube, so the interested reader can form his or her own opinion. The videos can be found by searching for "Issey Miyake Fashion Show: Women's Ready to Wear Autumn/Winter 2010."

In an interview Fujiwara explained how his collection was "an expression of space." From this statement, one gets the impression that the designers mainly used the mathematics as inspiration for their work, rather than creating an explicit illustration of the geometries. There was also a more concrete, somewhat poetic connection between the collection and the geometries. Before describing this connection, we need to introduce one more mathematical object: the *orbifold.*

An orbifold is a manifold with

singularities. Like manifolds, we can equip orbifolds with metrics modeled on a specific geometry. For our purposes, it should be sufficient to understand two related examples. Let's start with the two-dimensional case. A cone, with cone angle of $2\pi/3$, is an example of a Euclidean orbifold with one singular point of order three. This cone can be constructed in two ways. In a method similar to the construction of the torus, we can cut a wedge from the circle with angle $2\pi/3$ and tape up the sides. We could also cut just one slit from the edge of the circle to the center, and then roll up the disk so it wraps around itself three times. Mathematically, this process can be described as taking the quotient of the disk by a rotation. Away from the cone point, every point has a small neighborhood so that the metric looks just like a small disk in $\mathbb{R}^2$.

The higher dimensional analog of the cone can be constructed from a solid cylinder. Again, we can think about the construction in two ways: either as cutting a wedge and gluing opposite sides, or by this process of rolling up the cylinder so it wraps around itself three times. We see that in three-dimensional spaces our singular sets can be one dimensional. We have a whole line segment of singularities labeled with a three.

Photo by William Thurston

Scarves at the Peeling the Orange Reception.

The important point is that a particular three-dimensional orbifold can belong to at most one of the eight geometric classes, and that singular sets can be one dimensional. Sometimes these singular sets have several components, which are linked together. The three-sphere S^3, which is the set of points distance one from the origin in $\mathbb{R}^4$, is a three-dimensional manifold and is the model space for one of the eight geometries. However, for the *orbifold* S^3, with a one-dimensional singular set, the metric class depends on how the singular set is sitting inside S^3. A table of links is shown in figure 7. Each link corresponds to the orbifold S^3 with the given link as a singular set of order two. Each of the obifolds carries a different one of the eight geometries.

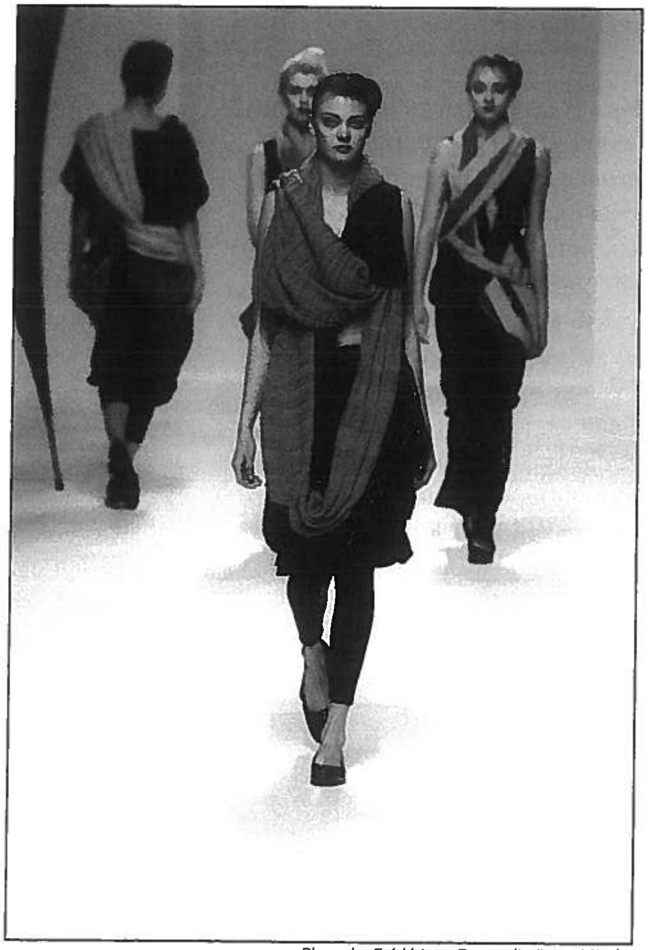


Photo by Frédérique Dumoulin/Issey Miyake

Issey Miyake's spring 2010 collection on the runway at Paris Fashion Week.

Thurston drew the links in figure 7. They were one of the many ideas that he shared with Fujiwara. The links intrigued Fujiwara, and they appeared as an integral part of several of the pieces in the fashion line, as seen on the models on the runway.

In an article written for the fashion magazine *Idoménée*, Thurston gave the following comment about the collection:

"The design team took these drawings as their starting theme and developed from there with their own vision and imagination. Of course it would have been foolish to attempt to literally illustrate the mathematical theory—in this setting, it's neither possible nor desirable. What they attempted was to capture the underlying spirit and beauty. All I can say is that it resonated with me." ■

Kelly Delp is an assistant professor at Buffalo State College specializing in geometric topology. In the spring of 2010 she visited Cornell, where she attended a joint math-fashion workshop that led to a project with Bill Thurston building surfaces.
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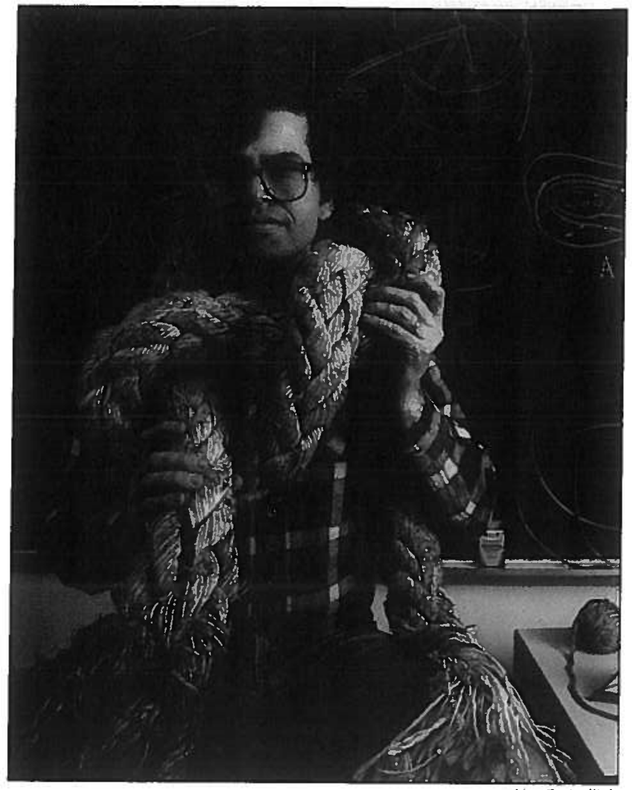
DOI: http://dx.doi.org/10.4169/mathhorizons.20.2.5

FROM THE EDITORS

To our readers:

On August 21, mathematics lost one of its giants, Bill Thurston.

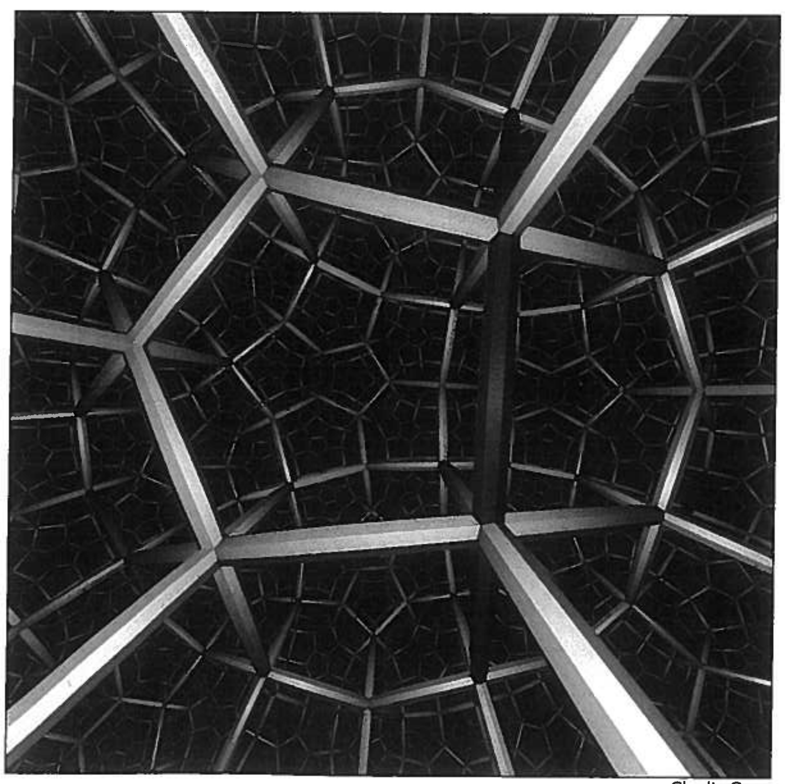
Best known for his celebrated *geometrization conjecture*, and proving it for a large class of manifolds, Thurston revolutionized the way that mathematicians understand three-dimensional space. Every compact three-dimensional manifold can be decomposed canonically along tori and spheres into simple pieces. Thurston's conjecture states that each of the simple pieces can be given one of eight homogeneous geometries. He won the Fields Medal in 1982 for this and other work in low-dimensional topology. Two decades later, his geometrization conjecture was famously proved in full by Grisha Perelman.



Abe Fraindlich

Bill Thurston.

In addition to his contributions to low-dimensional topology and geometry, Bill's passion for finding new ways to express and explain mathematical ideas was legendary. Whether creating movies, building 3D structures, or applying his ideas to novel realms such as fashion design (see p. 5), he embraced the firm conviction that mathematicians could do better in communicating the essence of their craft. His son Dylan said, "Bill emphasized constantly that the goal of mathematics, and the source of its beauty and utility, is *human understanding.* He aimed to gain and share an intuitive understanding of mathematics, recruiting all human senses—vision, motion, even touch."



Charlie Gunn

An image from the 1991 film *not Knot*.

It is fitting, then, that December will mark the opening of the U.S.'s first museum of mathematics. MoMath (see p. 14) will be a concrete realization of this idea: a place where people of all ages and backgrounds can experience and interact with mathematics, using at least most of their human senses.

In his 1994 article "On Proof and Progress in Mathematics," Bill reflected on his own place in the mathematical landscape: "I do think that my actions have done well in stimulating mathematics." We agree wholeheartedly.

BRUCE TORRENCE and STEPHEN ABBOTT
Editors

DOI: http://dx.doi.org/10.4169/mathhorizons.20.2.2