PROJECT 4

BUNGEE JUMPING

To the Instructor: You might want to wait until Chapter 5 is covered to assign this project.

http://www.idahoparks.org//Park_Pictures/Malad_Gorge/index.htm

FIGURE 1 The falls of the Malad River. Note the pedestrian bridge

Suppose that you have no sense. Suppose that you are standing on a bridge above the Malad River Canyon* and that you plan to jump off that bridge. See Figure 1. You have no suicide wish. Instead, you plan to attach a bungee cord to your feet, to dive gracefully into the void, and to be pulled back gently by the cord before you hit the river that is 174 feet below. You have brought a number of different cords to affix to your feet, including several standard bungee cords, a climbing rope, and a steel cable. You need to choose the stiffness and length of a cord to avoid the unpleasantness associated with an unexpected water landing. You are undaunted by this task, because you know math!

Each of the cords you have brought will be tied off so as to be 100 feet long when hanging from the bridge. Call the position at the bottom of the cord 0, and measure the position of your feet below that "natural length" as $x(t)$, where x increases as you go down and is a function of time t. See Figure 2. Then at the time you jump, $x(0) = -100$, and if your 6-foot frame hits the water head first, then at that time $x(t) = 174 - 100 - 6 = 68$.

You know that the acceleration due to gravity is a constant g, so the force pulling downwards on your body is mg. You know that when you leap from the bridge, air resistance will increase proportionally to your speed, providing a force in the opposite direction to your motion of about βv, where β is a constant and v is your velocity. Finally, you know that Hooke's law describing the action of springs says that the bungee cord will eventually exert a force on you proportional to your distance past the natural length of the cord. Thus you know that the force of the cord pulling you back from destruction can be expressed as

$$b(x) = \begin{cases} 0, & x \le 0 \\ -kx, & x > 0. \end{cases} \tag{1}$$

The number $k > 0$ in (1) is called the *spring constant* and is where the stiffness of the cord you use influences the equation. For example, if you used the steel cable, then k would be very large, giving a tremendous stopping force very suddenly as you passed the natural length of the cable. This could lead to discomfort, injury, or even a Darwin award. You want to choose the cord with a value of k large enough to stop you above or just touching the water but not too suddenly. Consequently, you are interested in finding the distance you fall below the natural length of the cord as a function of the spring constant. To do that, you must solve the differential equation that we have derived in words above: The net force mx'' on your body is given by

$$mx'' = mg + b(x) - \beta x'. \tag{2}$$

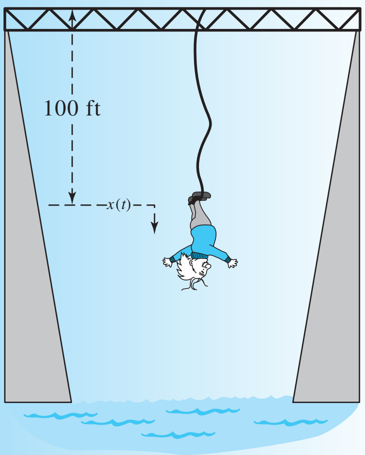

FIGURE 2 The fall from the bridge. As depicted here, $x(t) < 0$

Here mg is your weight, 160 pounds, and x' is the rate of change of your position below the equilibrium with respect to time—that is, your velocity. The constant β for air resistance depends on a number of things, including whether you wear your

*The Malad River Canyon is located in the Malad Gorge State Park near Hagerman, Idaho.

skin-tight pink Spandex or your skater shorts and XXL T-shirt, but you know that the value today is about 1.

Differential equation (2) is nonlinear, but inside it are two linear equations struggling to get out. You know how to solve such equations from your work in Chapters 4 and 5. When $x < 0$, the equation is $mx'' = mg - \beta x'$, while after you pass the natural length of the cord it is $mx'' = mg - kx - \beta x'$. You solve each equation separately and then piece the solutions together when $x(t) = 0$.

PROBLEM 1. Solve the equation $mx'' + \beta x' = mg$ for $x(t)$, given that you *step* off the bridge—that is, *no jumping, no diving*! "Stepping off" means that the initial conditions are $x(0) = -100$, $x'(0) = 0$. Use $mg = 160$, $\beta = 1$, and $g = 32$.

PROBLEM 2. Use the solution from Problem 1 to compute the length of time you free-fall (that is, the time it takes to go the natural length of the cord: 100 feet).

PROBLEM 3. Compute the derivative of the solution you found in Problem 1 and evaluate it at the time you found in Problem 2. You have found your downward speed when you pass the point where the cord starts to pull.

Problem 1 has given you an expression for your position t seconds after you step off the bridge, before the bungee cord starts to pull you back. Notice that it does not depend on the value of k. When you pass the natural length of the bungee cord, it does start to pull back, so the differential equation changes. Let t_1 denote the time you computed in Problem 2, and v_1 denote the speed you calculated in Problem 3.

PROBLEM 4. Solve the initial-value problem

$$mx'' + \beta x' + kx = mg, \quad x(t_1) = 0, \quad x'(t_1) = v_1.$$

For now you may use the value $k = 14$, but eventually you will need to replace this number with the values of k for the cords you brought. The solution $x(t)$ represents your position below the natural length of the cord after it starts to pull back.

Now you have an expression for your position as the cord pulls on your body. All you have to do is find the time t_2 at which you stop going down. When you stop going down, your velocity is zero—that is, $x'(t_2) = 0$.

PROBLEM 5. Compute the derivative of the expression you found in Problem 4 and solve for the value of t where the derivative is zero. Denote this time as t_2. Be careful that the time you compute is greater than t_1—there are several times when your motion stops at the top and bottom of your bounces! After you find t_2, substitute it back into the solution you found in Problem 4 to find your lowest position.

PROBLEM 6 (CAS). You have brought a soft bungee cord with $k = 8.5$, a stiffer cord with $k = 10.7$, and a climbing rope for which $k = 16.4$. Which, if any, of these cords can you use safely under the given conditions?

As you see, knowing a little bit of math is a dangerous thing. The assumption that the drag due to air resistance is linear applies only for low speeds. By the time you swoop past the natural length of the cord, that approximation is only wishful thinking, so your actual mileage may vary. Moreover, springs behave nonlinearly in large oscillations, so Hooke's law is only an approximation. Do not trust your life to an approximation made by a man who has been dead for two hundred years. Leave bungee jumping to the professionals.

STILL CURIOUS?

PROBLEM 7. You have a bungee cord for which you have not determined the spring constant k. To do so, you suspend a weight of 10 pounds from the end of the 100-foot cord, causing it to stretch 1.2 feet. What is the value of k for this cord?

PROBLEM 8 (CAS). What would happen if your 220-pound friend uses the bungee cord whose spring constant is $k = 10.7$?

PROBLEM 9 (CAS). If your heavy friend wants to jump anyway, then how short should you make the cord so that he does not get wet?

PROBLEM 10 (CAS). Graph the solutions you found in Problems 1 and 2 on the same coordinate axes. Explain the differences.