THEOREM $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots + \frac{1}{k^2} + \cdots = \frac{\pi^2}{6}$.

PROOF Euler began by introducing the function

$$f(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots$$

To Euler, $f(x)$ was just an infinite polynomial with $f(0) = 1$ (as is immediately apparent). Thus, it can be factored, in the manner developed above, provided we determine the roots of the equation $f(x) = 0$. To this end, observe that, for $x \neq 0$

$$\begin{aligned} f(x) &= x\left[\frac{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots}{x}\right] \\ &= \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots}{x} \\ &= \frac{\sin x}{x} \end{aligned}$$

by the Taylor Expansion of $\sin x$. Therefore, so long as x is not 0, solving $f(x) = 0$ amounts to solving $\frac{\sin x}{x} = 0$, which (through a simple cross-multiplication) reduces to solving $\sin x = 0$. As we have seen, the sine function equals 0 precisely for $x = 0$, $x = \pm\pi$, $x = \pm 2\pi$, and so on. But we must, of course, eliminate $x = 0$ from contention as a solution of $f(x) = 0$, since we have already noted that $f(0) = 1$. For the rest of the proof, see Dunham's *Journey through Genius*, pages $216 - 217$.