Concordia College

Math 105

Fall 2010

Exploring Mathematics

Samantha Briggs, Ellen Kramer, and Jessie Lenarz

Contents

1

Preliminaries

Before introducing new topics, a brief review of some pertinent information from arithmetic and algebra might be helpful. In the course of this book, we will often need to interpret words as mathematical symbols. Some common "translations" are

Mathematical Symbol	Word(s)
$+$	increased by, more than, combined, together, total of, sum, added to
$-$	decreased by, minus, less, difference between/of, less than, fewer than
$\times$	of, times, multiplied by, product of, increased/decreased by a factor of
$\div$	per, out of, ratio of, quotient of
$=$	is, are, was, were, will be, gives, yields, equals

1.1 Fractions

Many mathematical problems involve parts of objects rather than whole objects, so we will briefly review arithmetic with fractions. The top number of a fraction is known as the **numerator** and the bottom number of a fraction is known as the **denominator**. A fraction is in simplest form or lowest terms if the numerator and denominator have no common factors. To simplify a fraction, factor both the numerator and denominator and cancel any factors that occur in both.

Example 1.1. Simplify the following fractions.

(a) $\dfrac{15}{20}$ (b) $\dfrac{150}{650}$ (c) $\dfrac{36}{24}$

Solution. (a) $\dfrac{15}{20} = \dfrac{3 \times 5}{4 \times 5} = \dfrac{3}{4}$

(b) $\dfrac{150}{650} = \dfrac{3 \times 5 \times 10}{5 \times 10 \times 13} = \dfrac{3}{13}$

(c) $\dfrac{36}{24} = \dfrac{2 \times 2 \times 3 \times 3}{2 \times 2 \times 2 \times 3} = \dfrac{3}{2}$

■

To add or subtract fractions, the denominators must be the same. If they are not, we find a common denominator first. To find the least common denominator, factor the denominators completely. The least common denominator must include all factors that appear in either denominator to the highest power represented. For instance, the least common denominator for $\frac{1}{75}$ and $\frac{1}{18}$ is $2 \times 3^2 \times 5^2 = 450$ since $75 = 3 \times 5^2$ and $18 = 2 \times 3^2$. Once we have a common denominator, we rewrite the fractions so the values remain the same. This is done by multiplying both numerator and denominator by the same number (which is equivalent to multiplying the fraction by 1, so the value stays the same) that gives the denominator we desire. Once the denominators of the fractions are the same, we add (or subtract) the numerators and keep the common denominator. Finally, we reduce the fraction to lowest terms to arrive at the final answer.

Example 1.2. Add or subtract the following fractions.

(a) $\dfrac{1}{3} + \dfrac{1}{4}$ (b) $\dfrac{2}{15} + \dfrac{3}{20}$ (c) $\dfrac{4}{45} - \dfrac{7}{150}$

Solution. (a) The least common denominator is $3 \times 4 = 12$

$$\begin{aligned} \frac{1}{3} + \frac{1}{4} &= \frac{1 \times 4}{3 \times 4} + \frac{1 \times 3}{4 \times 3} \\ &= \frac{4}{12} + \frac{3}{12} \\ &= \frac{7}{12} \end{aligned}$$

(b) The least common denominator is $3 \times 4 \times 5 = 60$

$$\begin{aligned} \frac{2}{15} + \frac{3}{20} &= \frac{2 \times 4}{15 \times 4} + \frac{3 \times 3}{20 \times 3} \\ &= \frac{8}{60} + \frac{9}{60} \\ &= \frac{17}{60} \end{aligned}$$

(c) The least common denominator is $2 \times 3^2 \times 5^2 = 450$

$$\begin{aligned}\frac{4}{45} - \frac{7}{150} &= \frac{4 \times 10}{45 \times 10} - \frac{7 \times 3}{150 \times 3} \\ &= \frac{40}{450} - \frac{21}{450} \\ &= \frac{19}{450}\end{aligned}$$

■

Multiplication of fractions is straightforward. Multiply the numerators to find the numerator for the product and multiply the denominators to find the denominator of the product. In symbols,

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

To divide one fraction by another, we invert the second fraction and then multiply. In symbols,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c}$$

Cancelation of common factors before multiplying helps reduce the answer before you actually multiply. For example,

$$\frac{3}{10} \times \frac{5}{9} = \frac{\cancelto{1}{3}}{\cancelto{2}{10}} \times \frac{\cancelto{1}{5}}{\cancelto{3}{9}} = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

versus

$$\frac{3}{10} \times \frac{5}{9} = \frac{15}{90} = \frac{15}{15 \times 6} = \frac{1}{6}$$

Example 1.3. Multiply or divide the following fractions.

(a) $\frac{1}{3} \times \frac{4}{5}$ (b) $\frac{2}{15} \div \frac{7}{20}$ (c) $\frac{4}{45} \times \frac{9}{16}$

Solution. (a)

$$\begin{aligned}\frac{1}{3} \times \frac{4}{5} &= \frac{1 \times 4}{3 \times 5} \\ &= \frac{4}{15}\end{aligned}$$

(b)

$$\begin{aligned}
\frac{2}{15} \div \frac{7}{20} &= \frac{2}{15} \times \frac{20}{7} \\
&= \frac{2}{\cancelto{3}{15}} \times \frac{\cancelto{4}{20}}{7} \\
&= \frac{2}{3} \times \frac{4}{7} \\
&= \frac{2 \times 4}{3 \times 7} \\
&= \frac{8}{21}
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{4}{45} \times \frac{9}{16} &= \frac{\cancelto{1}{4}}{\cancelto{5}{45}} \times \frac{\cancelto{1}{9}}{\cancelto{4}{16}} \\
&= \frac{1}{5} \times \frac{1}{4} \\
&= \frac{1 \times 1}{5 \times 4} \\
&= \frac{1}{20}
\end{aligned}$$

■

Exercises

In Exercises 1-4, simplify the fraction.

1. $\frac{14}{21}$
2. $\frac{25}{300}$
3. $\frac{16}{64}$
4. $\frac{27}{81}$

In Exercises 5-8, find the least common denominator of the fractions.

5. $\frac{2}{15}, \frac{5}{9}$
6. $\frac{7}{12}, \frac{1}{45}$
7. $\frac{11}{42}, \frac{4}{35}$
8. $\frac{4}{39}, \frac{7}{15}$

In Exercises 9-16, perform the indicated operation.

9. $\dfrac{3}{5}+\dfrac{7}{9}$

10. $\dfrac{12}{13}-\dfrac{5}{26}$

11. $\dfrac{2}{75}+\dfrac{7}{45}$

12. $\dfrac{1}{3}-\dfrac{1}{8}$

13. $\dfrac{2}{3}\times\dfrac{9}{16}$

14. $\dfrac{1}{5}\div\dfrac{3}{10}$

15. $\dfrac{25}{36}\times\dfrac{18}{75}$

16. $\dfrac{15}{28}\div\dfrac{3}{7}$

1.2 Percentages

The word **percent** comes from the Latin *per centum* meaning per hundred. So if we say that 30% of students at Concordia participate in some type of music ensemble, that means that in a group of 100 Concordia students, 30 of them would be involved in a music ensemble.

Since percent is defined as per hundred, we can represent percents as fractions. 17% means 17 out of 100 or $\dfrac{17}{100}$. This fraction (and therefore the percent) can also be represented as a decimal. Note that $\dfrac{17}{100}=0.17$, so we have

$$17\% = \frac{17}{100} = 0.17$$

A simple rule for converting a percent to a decimal is to move the decimal point two places to the left.

Example 1.4. Write each of the following percents in decimal form.
(a) 53% (b) 234% (c) 0.05%

Solution. (a) Take the number 53 and move the decimal point two places to the left to arrive at 0.53.

(b) Take the number 234 and move the decimal point two places to the left to arrive at 2.34.

(c) Take the number 0.05 and move the decimal point two places to the left to arrive at 0.0005. ■

We can also convert decimal numbers to percents. The rule here is to move the decimal point two places to the right.

Example 1.5. Write each of the following decimals as a percent.
(a) 1.23 (b) 0.89 (c) 0.007

Solution. (a) Take the number 1.23 and move the decimal point two places to the right to arrive at 123%.

(b) Take the number 0.89 and move the decimal point two places to the right to arrive at 89%.

(c) Take the number 0.007 and move the decimal point two places to the right to arrive at 0.7%. ■

Recalling the "translations" discussed earlier, if we were to say "35% of 140 is 49" this would translate into the mathematical equation

$$35\% \times 140 = 49$$

or

$$0.35 \times 140 = 49$$

Example 1.6. Solve the following percent problems.

(a) What is 54% of 210?

(b) 25 is 40% of what number?

(c) 315 is what percent of 500?

Solution. (a) Let n be the unknown amount. Then the words translate into the equation

$$54\% \times 210 = n$$

So we have $n = 0.54 \times 210 = 113.4$

(b) Let n be the unknown amount. Then the words translate into the equation

$$25 = 40\% \times n$$

So we have

$$\begin{aligned} 25 &= 0.4 \times n \\ \frac{25}{0.4} &= n \\ 62.5 &= n \end{aligned}$$

(c) Let n be the unknown percent in decimal form. Then the words translate into the equation

$$315 = n \times 500$$

So we have

$$\begin{aligned} 315 &= n \times 500 \\ \frac{315}{500} &= n \\ 0.63 &= n \end{aligned}$$

So 315 is 63% of 500. ■

Taxes

One important application of percents is in sales tax. Minnesota has a 6.875% state sales tax. Food (not including prepared food, some beverages such as soda, and other items such as candy) and clothing are exempt from this tax.

Example 1.7. The other day I went to the Wal-Mart in Dilworth. I bought a t-shirt for $\$8.99$, a DVD for $\$19.99$, some cereal for $\$3.99$, and a candy bar for $\$0.75$.

(a) How much tax do I pay?

(b) What is my total bill?

Solution. (a) Since I am in Minnesota, I pay a 6.5% tax on the DVD and the candy bar – the t-shirt and cereal are exempt. The total amount I spent on the DVD and candy bar is

$$\$19.99 + \$0.75 = \$20.74$$

so the tax I pay is

$$\$20.74 \times 6.5\% = \$20.74 \times 0.065 = 1.3481$$

The tax would be rounded up to $\$1.35$.

(b) The total amount I pay will be the total of all the items and the tax I computed in part (a):

$$\$8.99 + \$19.99 + \$3.99 + \$0.75 + \$1.35 = \$35.07$$

■

Discounts

Many retailers use the idea of a percentage discount to entice shoppers to buy from them.

Example 1.8. A mattress sells regularly for $\$850$. It is offered on sale at 40% off.

(a) How much is the discount?

(b) What is the price after the discount?

(c) If the sales tax is 6.5%, what is the total cost of the mattress to the consumer?

Solution. (a) We compute 40% of $\$850$

$$40\% \times \$850 = 0.4 \times \$850 = \$340$$

(b) The price after the discount is the original price minus the discount

$$\$850 - \$340 = \$510$$

(c) The tax on the price after the discount is

$$6.5\% \times \$510 = 0.065 \times \$510 = \$33.15$$

so the total cost of the mattress is

$$\$510 + \$33.15 = \$543.15$$

■

Exercises

In Exercises 1-6, convert the following percents to decimal form.

1. 27%
2. 356%
3. 4%
4. 0.48%
5. 34.5%
6. 0.73%

In Exercises 7-12, convert the following decimals to percents.

7. 0.76
8. 2.78
9. 0.38
10. 0.005
11. 1.35
12. 0.00037

In Exercises 13-21, find the unknown quantity.

13. What is 45% of 112?
14. 7 is what percent of 110?
15. 35 is 3% of what number?
16. What is 62.4% of 87?
17. 15 is what percent of 28?
18. 71.8 is 14% of what number?
19. What is 97% of 38?
20. 91.6 is what percent of 215.3?
21. 16 is 27% of what number?
22. The state sales tax in Alabama is 4%. John Smith bought a Nintendo Wii priced at $\$225$.
    (a) What was the tax on the purchase?
    (b) What was the total price John paid?
23. The social security tax rate (deducted from your paychecks!) was 7.65% for 2007. Find the social security tax paid in 2007 for a person earning $\$34,000$ a year.
24. A jewelry store is selling rings at a 25% discount. If the original price of the ring was $\$600$, what is the price of the ring after the discount (and before tax)?

25. Some gas stations offer a discount on gas if you pay in cash. If you filled your tank and the pump registered $\$48.25$ and you pay cash to receive a 5% discount, how much do you pay for the gas?

26. A computer retailer buys a multimedia computer for $\$1,750$ and then sells it for $\$1,999$. What is the percent markup on the computer?

27. A new gas barbecue grill has been reduced for an end of summer sale by 25% to $\$478$. What was the original price of the grill?

28. A department store is having a "door-buster" sale. Merchandise is marked down 40% and if you shop between 8 am and noon, you receive an additional 10% off. Nick Clause is stocking up for Christmas gifts and buys a sweater at 10 am for his wife that is normally $\$55$. How much does he pay for the sweater before tax?

1.3 Logarithms

A logarithmic function with base a is the inverse function of the exponential function $y = a^x$, $a > 0$, $a \neq 1$,

$$\log_a x = y \Longleftrightarrow a^y = x$$

The cancelation rules are

$$\begin{aligned} \log_a(a^x) &= x, && \text{for all real numbers } x \\ a^{\log_a x} &= x, && \text{for all real numbers } x > 0 \end{aligned}$$

Laws of Logarithms: If a, x, y are positive and r is any real number

- $\log_a 1 = 0$
- $\log_a a = 1$
- $\log_a(xy) = \log_a x + \log_a y$
- $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
- $\log_a(x^r) = r \cdot \log_a x$
- $\log_a x = \dfrac{\log_b x}{\log_b a}$, $a \neq 1$

The special case when the base $a = e \approx 2.718$ is called the "natural logarithm"

$$\ln x = \log_e x$$

The special case when the base $a = 10$ is called the "common logarithm". Most scientific calculators have buttons for the natural and common logarithms. Logarithms are used in solving equations with the unknown in the exponent.

Example 1.9. Find the exact value of $\log_7 \dfrac{1}{49}$.

Solution.

$$\begin{aligned} \log_7 \frac{1}{49} &= \log_7 7^{-2} \\ &= -2 \end{aligned}$$

■

Example 1.10. Solve for x: $2^{x-4} = 17$

Solution.

$$\begin{aligned} 2^{x-4} &= 17 \\ \ln\left(2^{x-4}\right) &= \ln 17 \\ (x-4)\ln 2 &= \ln 17 \\ x - 4 &= \frac{\ln 17}{\ln 2} \\ x &= 4 + \frac{\ln 17}{\ln 2} \\ &\approx 8.0875 \end{aligned}$$

■

Exercises

In Exercises 1-6, find the exact value.

1. $\log_3 9$
2. $\log_{10} 0.01$
3. $\log_2 \frac{1}{2}$
4. $\log_5 0.2$
5. $\ln e$
6. $\log_4 64$

In Exercises 7-12, solve for x. Round answers to three decimal places

7. $2^{3x-1} = 10$
8. $e^{7x+3} = 10$
9. $1 - 3^{x+2} = 12$
10. $5^{2-x} = e^3$
11. $5^{x+4} = 4^{x+5}$
12. $2^{x-3} = 3^{x-2}$

1.4 Counting

This section deals with questions containing phrases such as "How many ...?" or "In how many ways ...?" or "Find the number of ...?" One way of answering this type of question is to simply list out all possibilities and count how many items are in your list. This approach is only practical for relatively short lists. For instance if you wanted to list all the possible ways for three people to line up, there are only 6 ways. But if we change the number of people to 5, there are 120 ways and if we have 10 people, there are 3,628,800.

If we do have a relatively small amount of things to count, one possible way to see all the possibilities is by using a tree diagram. A **tree diagram** is a visual aid that can be used to represent the outcomes of a particular situation. Tree diagrams can be used when one or multiple categories are present. A simple tree diagram involves a single category, for example, a situation in which you are drawing one golf ball from a bucket containing three golf balls: one red (R), one yellow (Y), and one green (G).

Drawing a One-Stage Tree Diagram: Starting from a single point,

1. Draw one branch for each possible outcome, and
2. Label each branch at its end to represent each outcome.

There are three outcomes R, Y, and G. So, there are three right-hand endpoints in the simple tree diagram in Figure 1.1, called a **one-stage tree diagram**, since one action has occurred.

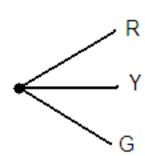


Figure 1.1: One-stage tree diagram

A **two-stage tree diagram** is used to represent situations in which two categories are being considered.

Drawing a Two-Stage Tree Diagram:

1. Construct a one-stage tree diagram for the outcomes of the first action. These branches are called the **primary branches**.
2. At the end of each primary branch, draw a one-stage tree diagram for each outcome of the second action. These branches are called the **secondary branches**.

For example, consider the same situation in which a person is drawing a golf ball from the bucket containing a red ball (R), a yellow ball (Y), and a green ball (G). After the first ball is drawn, the person gets to draw a second ball without replacing the first ball. The two-stage tree diagram in Figure 1.2 represents this situation.

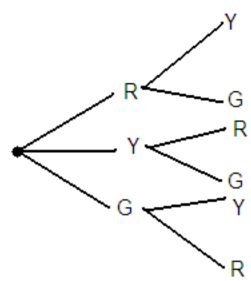


Figure 1.2: Two-stage tree diagram

This two-stage tree diagram shows that there are 6 possible outcomes possible since there are 6 right-hand endpoints in the diagram. The outcome corresponding to each endpoint is represented by a pair of letters in which the first letter represents the color of the first ball drawn and the second letter represents the color of the second ball drawn. For example, the outcome of choosing a red ball and then a yellow ball is represented by RY. The sample space contains the following outcomes: {RY,RG,YR,YG,GY,GR}.

Notice the total number of outcomes can be found by multiplying the number of primary branches by the number of secondary branches. Thus, in the above example, the total number of outcomes is $3 \times 2 = 6$. This idea is known as the Fundamental Counting Principle.

The **Fundamental Counting Principle** states that if an event or action A can occur in a ways, and, for each of these a ways, an event or action B can occur in b ways, the number of ways events or actions A and B can occur, in succession, is $a \times b$. The Fundamental Counting Principle can be generalized to situations that involve many categories.

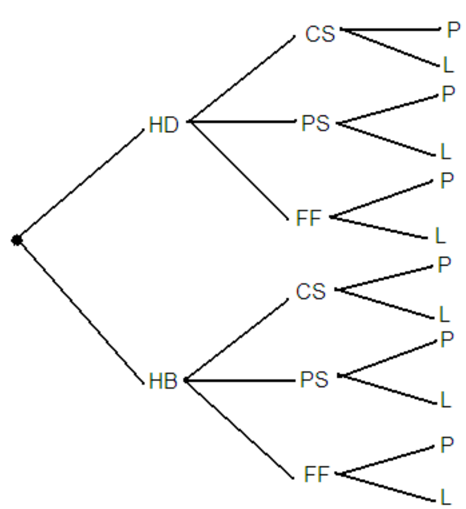
Example 1.11. Ron is at the deli getting a sandwich for lunch. He needs to decide what type of bread he wants (white or wheat), what kind of cheese he wants (swiss, cheddar, or pepperjack), and what kind of meat he wants (ham, turkey, roast beef, or chicken). Apply the Fundamental Counting Principle to find the number of possible sandwiches for Ron, assuming that he only chooses one type of bread, cheese, and meat.

Solution. Because 2 types of bread and 3 types of cheese are possible, there are $3 \times 2 = 6$ combinations of bread and cheese. Each of these 6 combinations can include any of the 4 kinds of meat, which gives us $6 \times 4 = 24$ different combinations for his sandwich. ■

We could have found the total number of possible combinations in one step by computing $2 \times 3 \times 4 = 24$. If we were to construct a tree diagram for this example, it would be a three-stage diagram, since there are three actions, or decisions, to be made. The first stage would have 2 branches corresponding to the two types of bread, the second would have 3 branches corresponding to the three types of cheese, and the third would have 4 branches corresponding to the four kinds of meat. The diagram would have 24 right-hand endpoints representing the 24 possible sandwiches.

Example 1.12. Consider a situation in which you are at a barbeque for a family reunion. For lunch, you get to choose between a hot dog (HD) or a hamburger (HB), coleslaw (CS), potato salad (PS), or french fries (FF), and pop (P) or lemonade(L). Construct a tree diagram for this experiment and determine the total number of possible lunches.

Solution. .



Since there are 12 right-hand endpoints on our tree diagram, there are 12 possible lunches in this situation. ■

One important fact to note is that the fundamental counting principle can be used only if the situation satisfies the uniformity criterion. A situation satisfies the **uniformity criterion** if the number of choices for a category is the same no matter which choices were selected for previous parts. In our previous example, no matter what you chose first (hamburger or hot dog) you still had three choices for your side, and no matter what side you chose, you still had two choices for your drink. Thus the uniformity criterion is satisfied.

Example 1.13. The Business Club has five members: Alex, Burt, Christine, Daniel, and Elizabeth. In how many ways can the club elect a president and a secretary if no one may hold more that one office and the secretary must be a man?

Solution. If we consider president first, there are five choice for president (no restrictions). However, we now have a problem because if a woman was selected president, there are three choice for secretary (the three men) but if a man was selected president, there are only two choices for secretary (the two men who were not selected president). Thus the uniformity criterion is not satisfied.

If we consider secretary first, there are three choices (the three men). No matter who is chosen as secretary, there are now four choices for president, so the uniformity criterion is satisfied. Thus we may use the fundamental counting principle and there are $3 \times 4 = 12$ possible ways to select a president and secretary. ∎

This suggests that whenever one or more parts of a situation have special restrictions, we should consider that part (or parts) before parts with no restrictions.

Counting: Permutations and Combinations

Recall the Fundamental Counting Principle from the previous discussion:

If an event or action A can occur in a ways, and, for each of these a ways, an event or action B can occur in b ways, the number of ways events or actions A and B can occur, in succession is $a \times b$.

Example 1.14. A password is used to protect your work computer account. It must consist of two upper-case letters followed by three digits. Determine how many different passwords are possible if

(a) repetition of letters and digits is permitted.

(b) repetition of letters and digits is not permitted.

Solution. (a) There are 26 letters and 10 digits $(0-9)$. Five positions must be filled.

$$\frac{26}{\mathsf{L}} \qquad \frac{26}{\mathsf{L}} \qquad \frac{10}{\mathsf{D}} \qquad \frac{10}{\mathsf{D}} \qquad \frac{10}{\mathsf{D}}$$

Since $26 \times 26 \times 10 \times 10 \times 10 = 676,000$, there are $676,000$ different possible arrangements.

(b) There are 26 possibilities for the first position. But since repetition is not allowed, there are only 25 possibilities for the second position. For the same reason, the number of digits also change.

$$\frac{26}{L} \qquad \frac{25}{L} \qquad \frac{10}{D} \qquad \frac{9}{D} \qquad \frac{8}{D}$$

Since $26 \times 25 \times 10 \times 9 \times 8 = 468,000$, there are $468,000$ different possible arrangements. ■

A **permutation** is any ordered arrangement of a given set of objects. "Red, Orange, Yellow" and "Yellow, Red, Orange" are two different permutations of the same three colors. When we are determining the number of possible permutations, we assume that repetition of an item is not permitted. The number of permutations of n distinct items is n factorial, represented as $n!$, where

$$n! = n(n-1)(n-2)\ldots(3)(2)(1)$$

We will define $0! = 1$.

Example 1.15. A teacher is trying to line up 8 children. How many different arrangements are possible?

Solution. Since there are 8 children, the number of permutations is $8!$.

$$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40,320$$

The 8 children can be arranged $40,320$ different ways. ■

We don't always want to find the different arrangements of all of the objects we have. Sometimes we only want to use a few of the objects. The number of permutations possible when r objects are selected from n objects is found by using the **permutation formula**

$${}_nP_r = \frac{n!}{(n-r)!} = n \times n-1 \times n-2 \times \cdots \times n-(r-1)$$

Example 1.16. The 9 members of the yearbook club want to randomly select a president, vice president, and secretary by placing all members' names in a hat and drawing randomly. How many different arrangements or permutations of officers are possible?

Solution. There are 9 members, so $n = 9$. Three officers are to be selected, so

$r = 3$. Thus,

$$
\begin{aligned}
{}_9P_3 &= \frac{9!}{(9-3)!} \\
&= \frac{9!}{6!} \\
&= \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} \\
&= \frac{9 \times 8 \times 7 \times 6!}{6!} \\
&= 9 \times 8 \times 7 \\
&= 504
\end{aligned}
$$

Therefore, with nine members there are 504 different arrangements for president, vice president, and secretary. ■

So far, all the examples we have discussed have involved arrangements with distinct items. However, we can also consider permutations in which some objects are duplicates. For instance, the word MOM contains three letters, of which the two Ms are duplicates. How many permutations of the letters in the word MOM are possible? If the two Ms were distinguishable in some way, say M_1 and M_2, there would be six permutations.

$$
\begin{array}{ccc}
M_1OM_2 & M_1M_2O & OM_1M_2 \\
M_2OM_1 & M_2M_1O & OM_2M_1
\end{array}
$$

However, if the Ms are not distinguishable, we see there are only three permutations

$$MOM \quad MMO \quad OMM$$

The number of distinct permutations of n objects where n_1 of the objects are identical, n_2 of the objects are identical, ..., n_r of the objects are identical is found by

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

Example 1.17. In how many different ways can the letters of the word "BUMBLEBEE" be arranged?

Solution. Of the 9 letters, three are Bs, three are Es, one is an L, one is an M, and one is a U. The number of possible arrangements are

$$\frac{9!}{3!3!1!1!1!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 3 \times 2 \times 1 \times 1 \times 1 \times 1} = 9 \times 8 \times 7 \times 5 \times 4 = 10,080$$

■

The key thing to remember is we are dealing with permutations if the *order of the selected items matters*. If the order of the selected items is not important to the results, the problem is a combination problem. Recall that $\{a, b, c\}$ and $\{b, a, c\}$ are two different permutations because of the different order. A **combination** is a distinct group (or set) of objects without regard to their arrangement. Therefore $\{a, b, c\}$ and $\{b, a, c\}$ represent the same combination but $\{a, b, c\}$ and $\{b, d, e\}$ do not.

Example 1.18. Determine whether the following problems are a permutation or combination problem.

(a) Six friends have decided to form a club and must elect a president and vice president. How many different ways can the president and vice president be selected?

(b) Of the six friends, two will be attending a conference together. How many different ways can they do so?

Solution. (a) Since the position of president is different from that of the vice president, we have a permutation problem. If A was selected as president and B was selected as vice president, that would be a much different situation than if B were selected as president and A was selected as vice president. Thus, order of the selection matters.

(b) Since the order in which the two individuals are selected to go to the conference does not matter, we have a combination problem. It does not matter if A was selected and then B was selected or if B was selected and then A was selected. ■

The number of combinations possible when r objects are selected from n objects is determined by the **combination formula**

$$_nC_r = \frac{n!}{(n-r)!r!}$$

Example 1.19. Laura has bought 8 paintings but only has room to hang up 5 at this time. How many different arrangements can she hang up?

Solution. Since the order in which the 5 paintings are selected does not matter, this is a combination problem. There are a total of 8 paintings, so $n = 8$. She must select 5 paintings to hang up, so $r = 5$.

$$\begin{aligned} _8C_5 &= \frac{8!}{(8-5)!5!} \\ &= \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1} \\ &= 56 \end{aligned}$$

■

Example 1.20. The Larson family decided to try a new restaurant for dinner. The restaurant offered a three-course meal consisting of courses A, B, and C. Course A has 3 options, course B has 5 options, and course C has 2 options. If the kids have been told by their parents to choose 2 items from course A, 3 items from course B, and 1 item from course C, how many different dinner combinations are possible?

Solution. For course A, 2 of the 3 items must be selected, which is represented as ${}_3C_2$. For course B, 3 of the 5 items must be selected, which is represented as ${}_5C_3$. For course C, 1 of the 2 items must be selected, which is represented as ${}_2C_1$.

$$ {}_3C_2 = 3 \qquad\qquad {}_5C_3 = 10 \qquad\qquad {}_2C_1 = 2 $$

We will use the counting principle to determine the total number of dinner combinations by multiplying the number of choices from course A, B, and C:

$$\begin{aligned}\text{Total number of dinner choices} &= {}_3C_2 \times {}_5C_3 \times {}_2C_1 \\ &= 3 \times 10 \times 2 \\ &= 60\end{aligned}$$

Thus, there are 60 different possible combinations in this particular situation. ■

Exercises

For Exercises 1 and 2, construct tree diagrams for the given situations. List all possible outcomes and record the total number of outcomes possible.

1. A jar holds five marbles: a red one, a black one, a blue one, a green one, and a yellow one. Jennifer draws one marble from the jar. Without replacing that marble, she draws a second marble.
2. You are at a family barbeque for lunch. You get to choose if you want a bratwurst (BW) or a cheeseburger (CB). Then you get to choose if you want coleslaw (CS) or potato salad (PS). For dessert you have the choice between a cookie (C), a brownie (B), or jello (J). Lastly, to drink, you have the choice between lemonade (L) or pop (P).
3. When you went to the bank to receive a new debit card, you also had to choose your new security code to use that debit card in an ATM. The security code must be all numbers, and is four digits long. How many different security codes are possible?
4. You must create a new password for your e-mail account. Due to security reasons, this password must be exactly seven characters – two lower-case letters followed by five numerals. How many different passwords are possible?
5. Mrs. Berndt's class of 22 kids are getting ready to head to gym. Mrs. Berndt wants them to line up in a single file line. How many different ways can the 22 kids be lined up?

6. At a local track meet, 12 people are running in the 200-meter event. The top 8 participants will qualify for the state track meet. How many different ways are there for the participants to qualify for state?

7. In how many ways can the letters in the word "MATH" be arranged?

8. In how many ways can the letters in the word "MISSISSIPPI" be arranged?

9. Determine whether the following situations represent a permutation or combination problem.

   (a) The hockey team is selecting two captains, who will work as co-captains, from their 22-player team. In how many different ways can they do so?

   (b) The foreign language club is selecting three officers: president, vice president, and secretary, from their 15-member club. In how many different ways can the three officers be selected?

10. Jan has 16 different flowers from which she can make a bouquet. However, only 12 of the flowers will fit in the vase that she wants to use. How many different ways can she choose which flowers to use in the vase?

11. A couple is planning the menu for their upcoming holiday party. They must choose the appetizer, the main course, and the dessert. The restaurant advises them to choose 5 of the 8 appetizers, 3 of the 5 main courses, and 3 of the 7 desserts. How many different dinner combinations are possible?

2

Taxicab Geometry

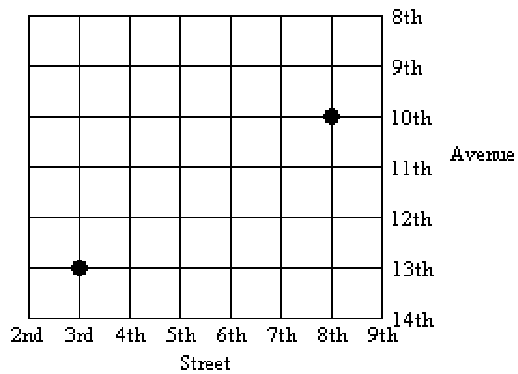
What do you think of when someone says geometry? Perhaps you think of points and lines, congruence of triangles, the Pythagorean Theorem, or an endless list of axioms, postulates and theorems that one must prove. In almost all cases, what you are thinking of is called Euclidean geometry, where points are plotted in the coordinate plane, any two points determine a line, and the distance between two points is the length of the line segment connecting those two points. Would it surprise you to know that non-Euclidean geometries exist?

There are many examples of non-Euclidean geometries. Spherical geometry is the geometry of the two dimensional surface of a sphere. It has many applications in astronomy and navigation. Spherical geometry is a geometry in which two points do not determine a line, but rather an arc. Hyperbolic geometry and elliptical geometry are examples of geometries in which the parallel postulate is no longer true. The parallel postulate of Euclidean geometry states that given a line ℓ and a point A there is exactly one line through A that does not intersect ℓ. In hyperbolic geometry, there are an infinite number of lines through A that do not intersect ℓ. In elliptic geometry, all lines through A intersect ℓ.

Each of the aforementioned geometries requires an extensive background of Euclidean geometry to work with. One nice example of a non-Euclidean geometry that does not require an extensive knowledge of Euclidean geometry is taxicab geometry. Taxicab geometry is a non-Euclidean geometry in which points are plotted in the coordinate plane and two points determine a line, but distance is measured in a different way.

2.1 Taxicab Distance

Exploration 2.1. Suppose, in the city shown below, that we want to ride in a taxicab along city streets from the corner of 8th Street and 10th Avenue to the corner of 3rd Street and 13th Avenue.



(a) How many blocks does it take to make such a trip?

(b) Does every route in the city grid from the corner of 8th Street and 10th Avenue to the corner of 3rd Street and 13th Avenue take the same distance?

(c) Does every route in the city grid from the corner of 8th Street and 10th Avenue to the corner of 3rd Street and 13th Avenue that continues to make progress at every point take the same distance?

The taxicab distance from point A to point B is the minimum number of blocks a taxicab would have to drive to get from point A to point B. The formal mathematical definition of the taxicab distance from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

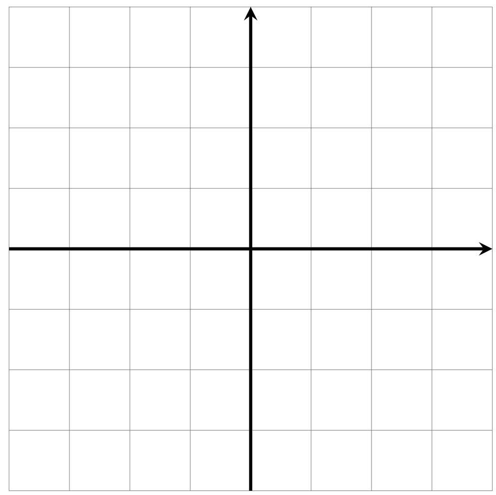
$$d_T(A, B) = |x_1 - x_2| + |y_1 - y_2|$$

Notice how this differs from the definition of the Euclidean distance from $A(x_1, y_1)$ to $B(x_2, y_2)$

$$d_E(A, B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Is there a relationship between these two measures of distance? Let's investigate.

Exploration 2.2. (a) Graph the points $A = (1, 3)$, $B = (1, -2)$, $C = (-3, -1)$, and $D = (0, 3)$.



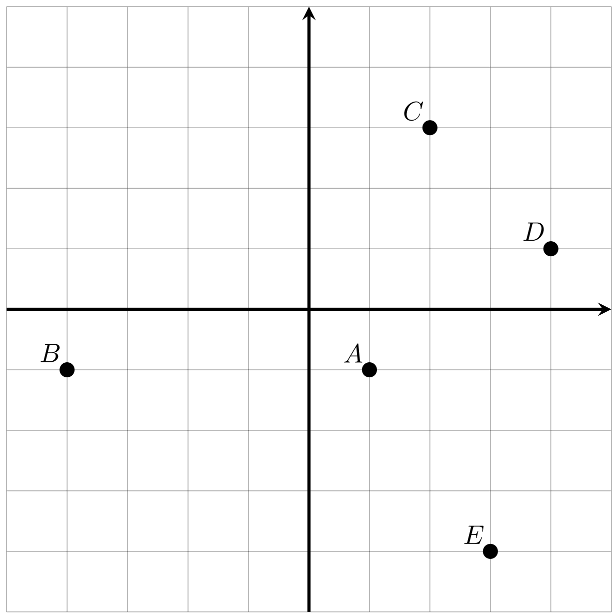
(b) Now find the following distances in both Euclidean and taxicab geometries. Give a decimal approximation to 2 decimal places.

	Euclidean distance	Taxicab distance
from A to B		
from B to C		
from C to D		

(c) If you know the Euclidean distance between two points, does that tell you what the taxicab distance is? Why or why not?

(d) If you know the taxicab distance between two points, does that tell you what the Euclidean distance is? Why or why not?

Exploration 2.3. (a) Consider the points in the following graph:

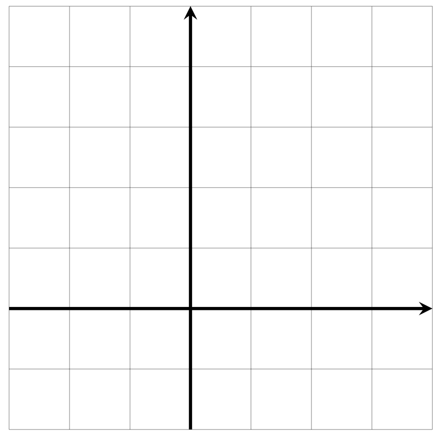


Calculate the following distances in both Euclidean and taxicab geometries. Give a decimal approximation to 2 decimal places.

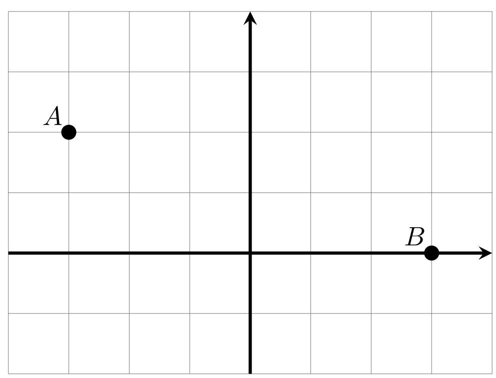
	Euclidean distance	Taxicab distance
from A to B		
from A to C		
from A to D		
from A to E		

(b) Is the Euclidean distance between two points always less than or equal to the taxicab distance? If so, explain why. If not, give an example where the Euclidean distance is greater than the taxicab distance.

Exploration 2.4. One night the 911 dispatcher for Taxicab City receives a report of an accident at $X = (-1, 4)$. There are two police cars in the area, car C at $(2, 1)$ and car D at $(-1, -1)$. Which car should be sent to the scene of the accident to arrive most quickly? (Since the cars must drive on the streets, we use taxicab geometry to measure distances.)



Exploration 2.5. Using taxicab geometry, consider the points $A = (-3, 2)$ and $B = (3, 0)$.



(a) Is the point $(-2, -3)$ closer to A or to B?

(b) Is the point $(1, -2)$ closer to A or to B?

(c) Find one point that is exactly the same distance from A is it is from B. Mark it on the graph.

(d) Find another such point. Mark it on the graph.

(e) Mark all points on the graph that are equally distant from A and from B. (Remember, this includes points with non-integer coordinates.)

Exercises

Complete your work on separate paper and turn it in. All taxicab pictures should be completed on graph paper. Do not put different problems on the same graph. (For example, problems 4(a) and 4(b) may be done on the same graph, but not problems 4 and 5.)

1. (a) On a single *large* graph, plot the following points:

$$\begin{array}{ll} A = (5,4) & B = (1,2) \\ C = (4,-3) & D = (-1,5) \\ E = (-5,-4) & F = (1,-2) \end{array}$$

   (b) Find the Euclidean distance between A and B.

   (c) Find the taxicab distance between A and B.

   (d) Find the Euclidean distance between B and F.

   (e) Find the taxicab distance between B and F.

   (f) Find the Euclidean distance between F and C.

   (g) Find the taxicab distance between F and C.

2. Let $A = (1,3)$. Find a point B such that the taxicab distance between A and B is 4, but the Euclidean distance from A to B is *not* 4.

3. Let $C = (1,0)$.

   (a) Find five different points that are a taxicab distance of 5 from C, but are *not* a Euclidean distance of 5 from C.

   (b) Graph *all* of the points that are a taxicab distance of 5 from C, including those that are a Euclidean distance of 5 from C.

   (c) Come up with a good name for the answer to part 3b.

4. Let $A = (3,1)$ and $B = (7,1)$.

   (a) Find a point C so that the taxicab distance between A and C is the same as the taxicab distance between B and C.

   (b) Find a different point D so that the taxicab distance between A and D equals the taxicab distance between B and D.

5. Let $A = (-1,1)$ and $B = (3,3)$.

   (a) Find a point C so that the taxicab distance between A and C is the same as the taxicab distance between B and C.

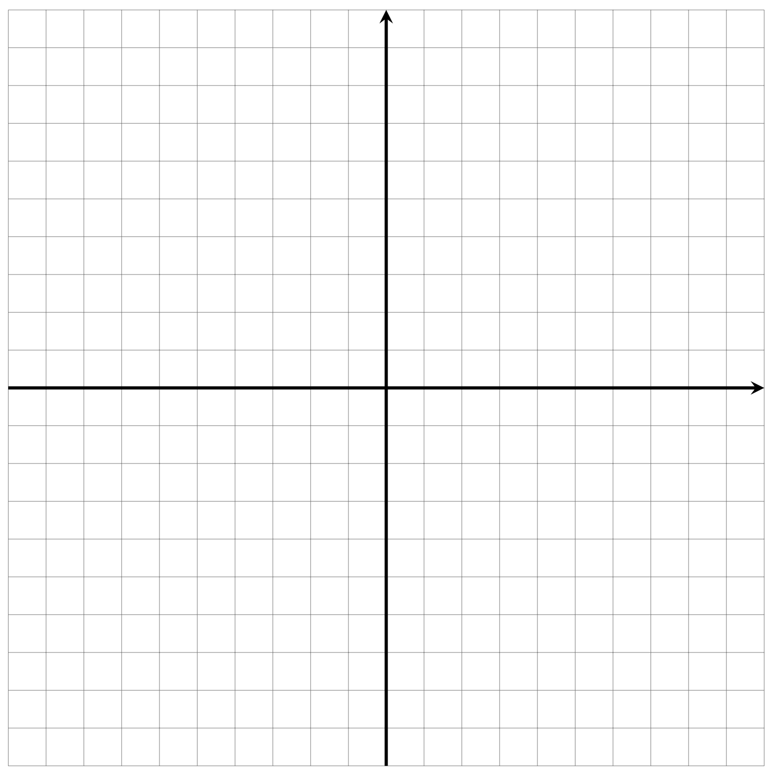
   (b) Find a different point D so that the taxicab distance between A and D equals the taxicab distance between B and D.

2.2 Taxicab Circles

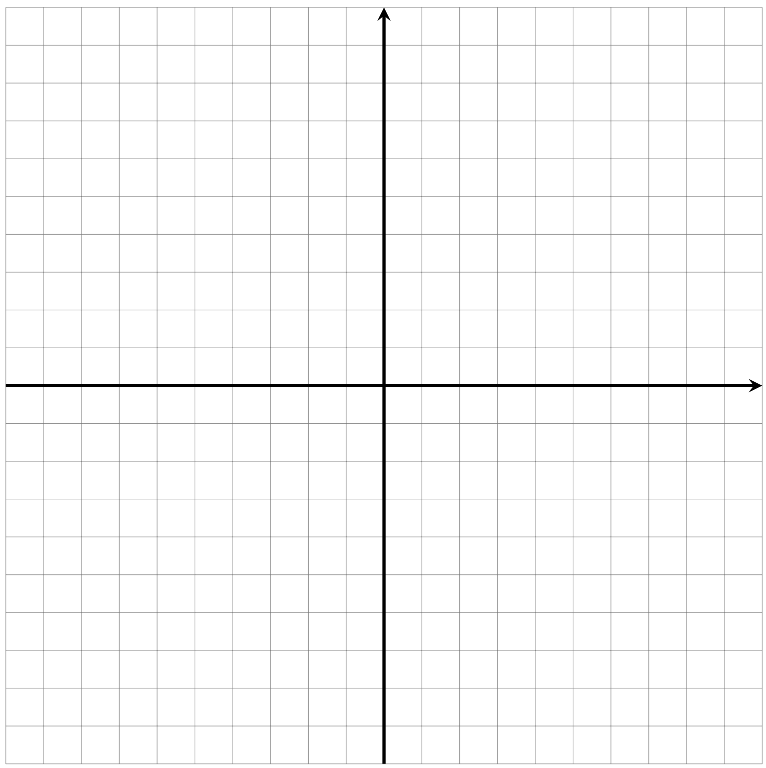
A circle is the set of all points equidistant from a fixed point (called the center). The distance from the center of a circle to any point on the circle is called the radius of the circle. A taxicab circle is a circle where the radius is measured using taxicab distance. Put another way, the taxicab circle with radius r and center C is the set of all points at distance r from C. In symbols, the taxicab circle with radius r and center C is

$$\{P : d_T(C, P) = r\}$$

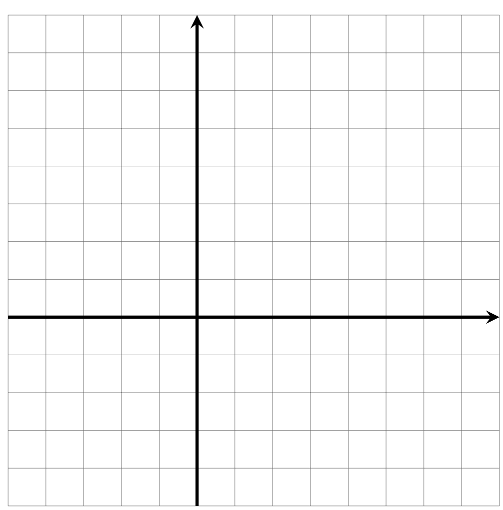
Exploration 2.6. Draw the taxicab circle of radius 5 around the point $P = (3, 4)$.



Exploration 2.7. Draw the taxicab circle of radius 6 around the point $Q = (2, -1)$.



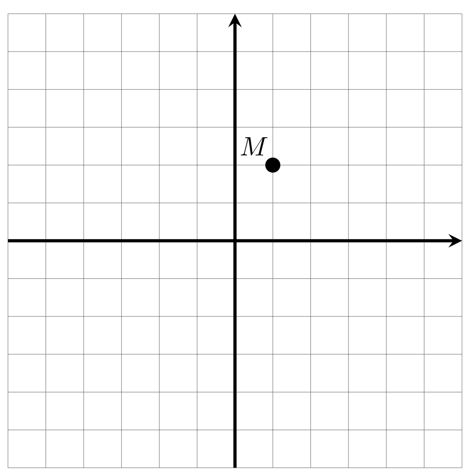
Exploration 2.8. On a single graph, draw taxicab circles around point $R = (1, 2)$ of radii 1, 2, 3, and 4.



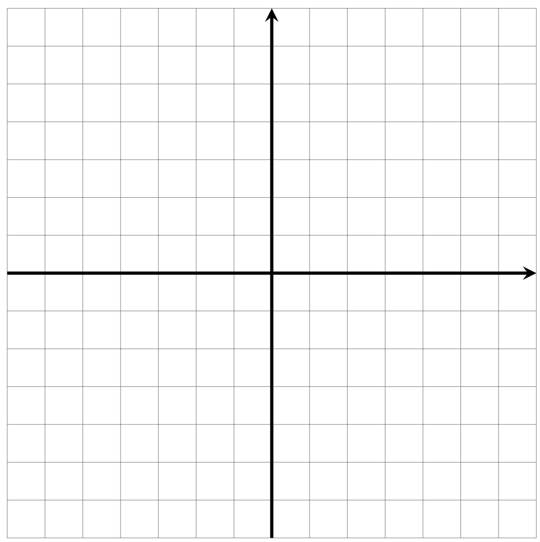
Exploration 2.9. Describe a quick technique for drawing a taxicab circle of radius r around a point P.

Exploration 2.10. What is a good value for π in taxicab geometry? Consider the definition of π as it relates to circumference of a circle.

Exploration 2.11. George works in Taxicab City for the 3M plant, located at $M = (1, 2)$. He goes out to eat for lunch once a week, and out of company loyalty, he likes to walk exactly 3 blocks from the plant to do so. Where in the city are restaurants at which George can eat? Draw their locations on the graph.

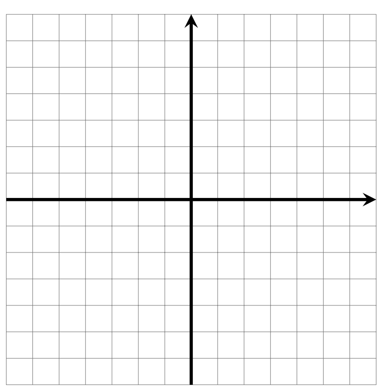
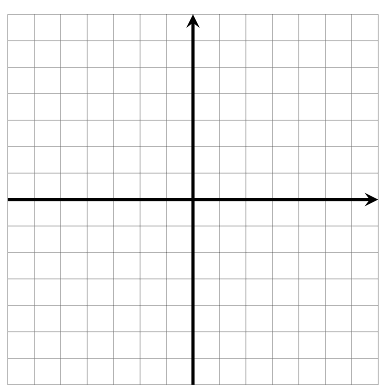


Exploration 2.12. In our taxicab city, a builder wants to construct an apartment building within six blocks of the mall at $M = (-2, 1)$ and within four blocks of the tennis courts at $T = (3, 3)$. Where can the builder build?

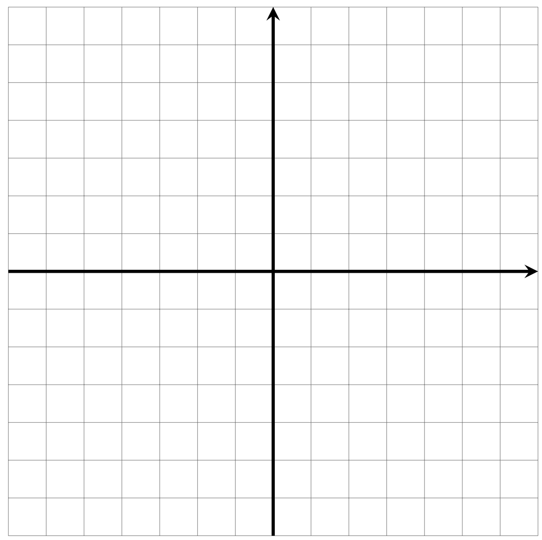


Exploration 2.13. Fred Finnegan campaigned to become mayor of Taxicab City on a platform of installing drinking fountains throughout the city, so that no one would ever be more than three blocks from a free drink of water. He won the election, but since his predecessor depleted the city treasury, he needs to spend money judiciously. Suppose the city is 14 blocks square, as shown in the grids below.

(a) Come up with at least 2 different plans for where to locate the drinking fountains. (If you need more room to practice, there are more grids on the next page.)

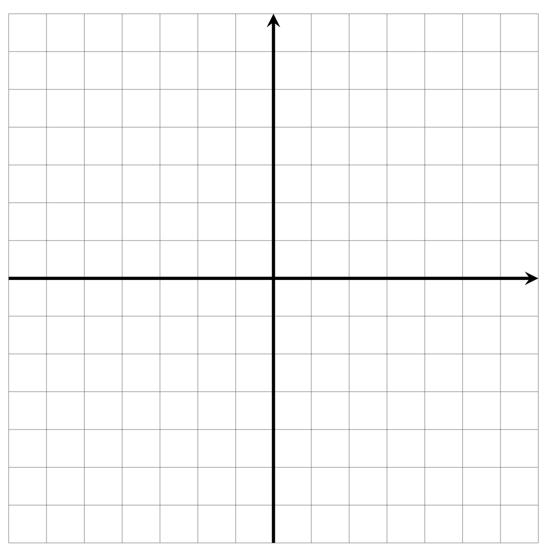



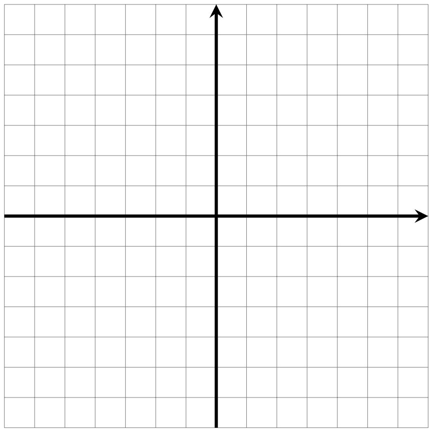
(b) Come up with the most cost-efficient way for the mayor to fulfill his campaign promise. (That is, give a map of where to locate the drinking fountains.)

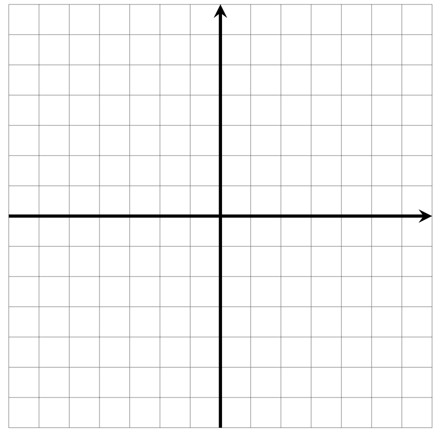


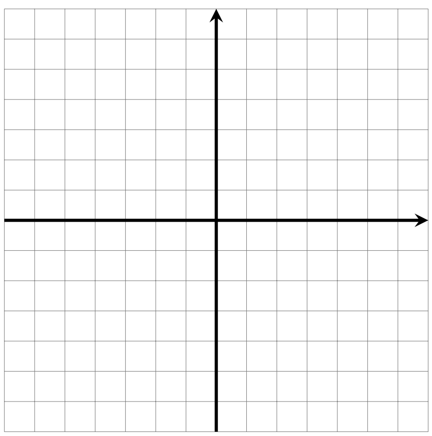
(c) Suppose Taxicab City were to expand into the surrounding countryside as the population grows. Describe how to extend your pattern indefinitely into the new, surrounding blocks.

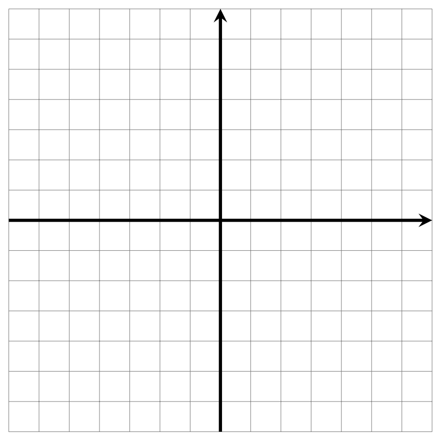
Exploration 2.14. In the same 14×14 grid, the city decided to install fire hydrants so that every resident is within 4 blocks of a fire hydrant. What is the fewest number of hydrants needed, and where should they be located?

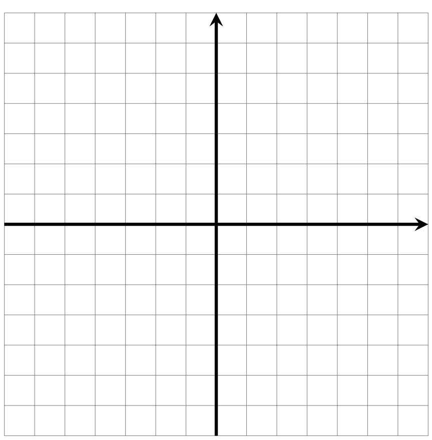


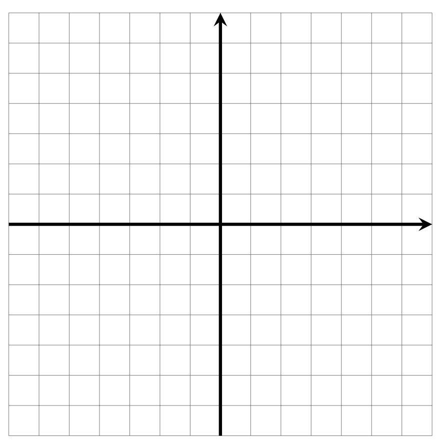












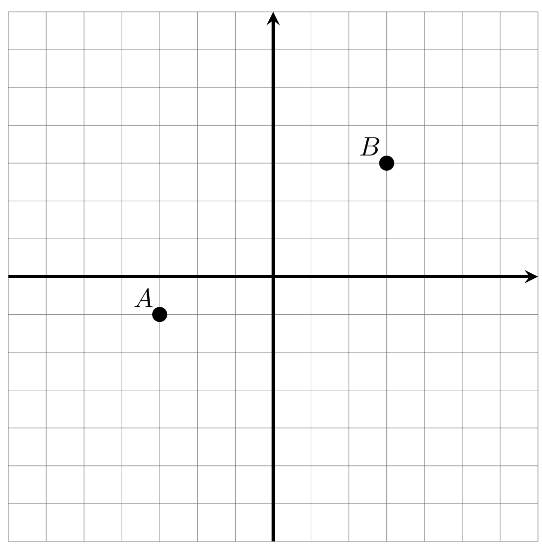
Exercises

Complete your work on separate paper and turn it in. All taxicab pictures should be completed on graph paper. Do not put different problems on the same graph. (For example, problems 4(a) and 4(b) may be done on the same graph, but not problems 4 and 5.)

1. Graph all of the points that are a taxicab distance of exactly 4 from the point $A = (-2, -1)$.

2. (a) Graph the taxicab circle that is centered at $(1, 4)$ with a radius of 3.
   (b) Graph the taxicab circle that is centered at $(-1, -3)$ with a radius of $\frac{5}{2}$.

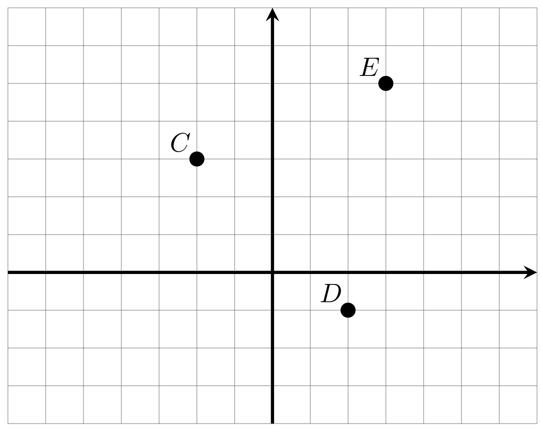
3. Alice and Bob reside in Taxicab City, which is laid out like a perfect grid centered on $(0, 0)$. North-south and east-west streets join every point with integer coordinates. Alice works at an art school located at $A = (-3, -1)$. Bob works in a bakery located at $B = (3, 3)$. Alice and Bob just got married and are looking for a house in the city. Alice has always dreamed of a cozy little house on a corner lot, so they will only consider houses located at street corners. Because they will walk to their jobs along the city streets, they measure all their distances using taxicab geometry.



   (a) Is it possible for Alice and Bob to live exactly 5 blocks from A and exactly 4 blocks from B? If so, find all locations that work and name their coordinates. If not, why is it impossible?
   (b) Is it possible for Alice and Bob to live exactly 8 blocks from A and exactly 6 blocks from B? If so, find all locations that work and name their coordinates. If not, why is it impossible?

(c) Is it possible for Alice and Bob to live 8 or fewer blocks from A and 6 or fewer blocks from B? If so, find all locations that work and shade them in on a graph. If not, why is it impossible?

4. Charles and Denise also live in Taxicab City. Charles works at the Chevrolet dealership at $C(-2, 3)$ and Denise works at the donut shop at $(2, -1)$.



(a) Is it possible for Charles and Denise to live within 5 blocks of C and 4 blocks of D? If so, find all locations that work and shade them in on a graph. If not, why is it impossible?

(b) Charles and Denise have a daughter, Elizabeth, who goes to the elementary school at $E(3, 5)$. Is it possible for the family to live within 5 blocks of C, 4 blocks of D, and 3 blocks of E? If so, find all locations that work and shade them in on a graph. If not, why is it impossible?

(c) Is it possible for the family to live within 5 blocks of C, 4 blocks of D, and 2 blocks of E? If so, find all locations that work and shade them in on a graph. If not, why is it impossible?

(d) Is it possible for the family to live within 5 blocks of C, 4 blocks of D, and 1 block of E? If so, find all locations that work and shade them in on a graph. If not, why is it impossible?

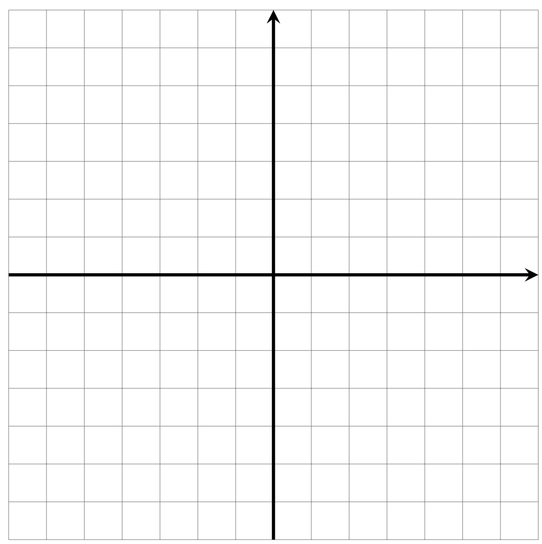
5. (a) The city council of Taxicab City has decided to build parks so that every resident of Taxicab City is within 6 blocks of a park. If Taxicab City is currently 14 blocks square, what is the minimum number of parks needed and where should they be located?

(b) Suppose the city already has a park located at $(1, 2)$. Does that change the minimum number of parks needed and where they are located? If it does, what is the new minimum number and where are they located? If it doesn't change, why not?

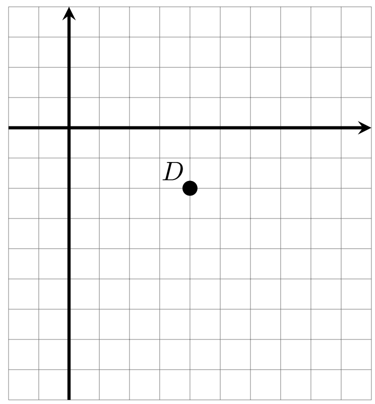
2.3 Taxicab Applications

Taxicab geometry is a more useful model of urban geography than Euclidean geometry. In an urban setting, people travel along streets rather than as the crow flies. Note that taxicab geometry assumes that all streets are laid out north/south or east/west, streets have no width, and buildings are points with no base area. These assumptions are obviously not true in most urban settings, but even with these simplifying assumptions, taxicab geometry is very useful in solving problems in urban geography.

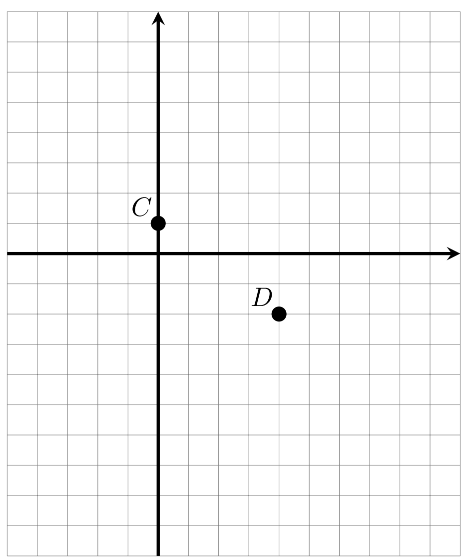
Exploration 2.15. A furniture company in our taxicab city stores unfinished tables in their warehouse at $W = (-3, 2)$. They will ship their tables to their factory F for finishing, and then from there to their retail store at $S = (4, 0)$. If they want to minimize the total distance they ship the tables, where can they put their factory F? Draw all possible locations for F on the graph below.



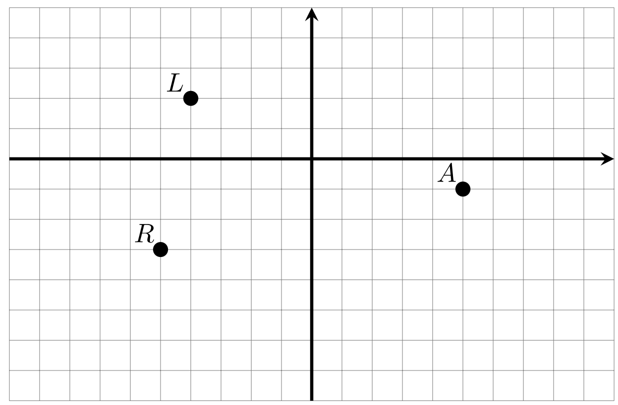
Exploration 2.16. (a) Doug moves to Taxicab City and works at the distillery at $D = (4, -2)$. He walks to work along the city blocks. Because of a heart condition, Doug cannot live more than 5 blocks from work. On a graph, shade in all the places Doug can live.



(b) On second thought, Doug realizes that he also wants to live near the church at $C = (0, 1)$. He is looking for an apartment A so that the distance from A to C *plus* the distance from A to D is at most 9 blocks. Shade in all the places Doug can live.



Exploration 2.17. Acme Industrial Parts wants to build a factory in Taxicab City. It needs to receive shipments from the railroad depot at $R = (-5, -3)$ and ship parts out by plane, so it wants the factory to be located so that the total distance from the depot to the factory to the airport at $A = (5, -1)$ is at most 16 blocks. However, a city noise ordinance prohibits any factories from being built within 3 blocks of the public library at $L = (-4, 2)$. Where can Acme build its factory?

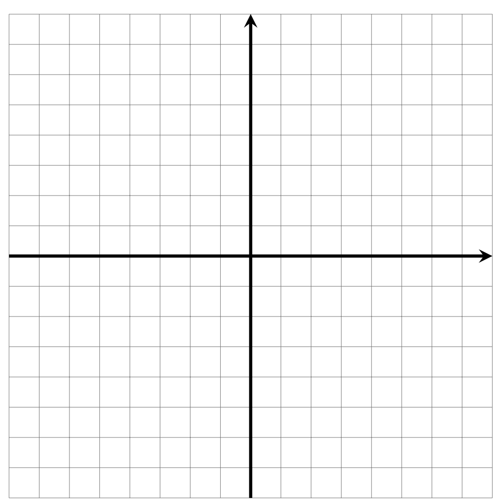


An ellipse is the set of all points whose distances to two fixed points sum to a constant. The two fixed points are called the foci (singular: focus). A taxicab ellipse is an ellipse where distance is measured using the taxicab distance. In symbols, a taxicab ellipse with foci F_1 and F_2 is

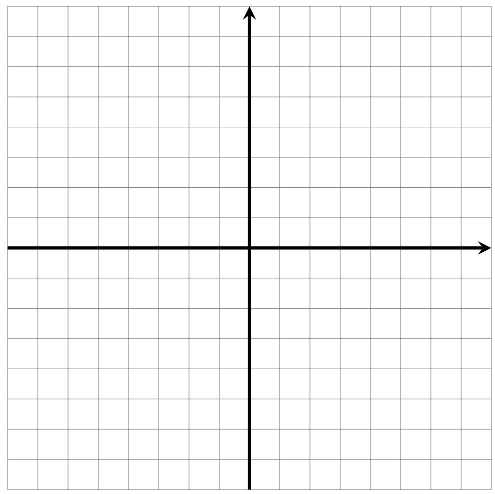
$$\{P : d_T(P, F_1) + d_T(P, F_2) = k\}$$

where k is some constant.

Exploration 2.18. (a) Draw the taxicab ellipse with foci $(-2, 1)$ and $(4, 1)$, so that the total distance from each point to the foci is 8.

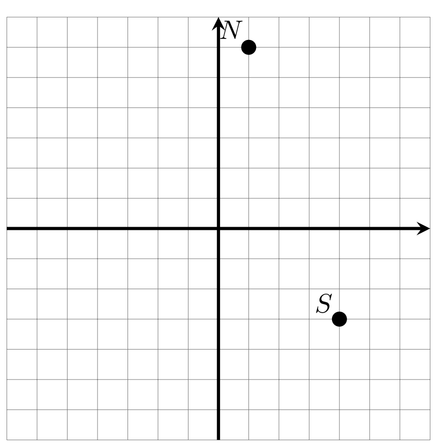


(b) Draw the taxicab ellipse with foci $(-3,-2)$ and $(4,3)$, so that the total distance from each point to the foci is 16.

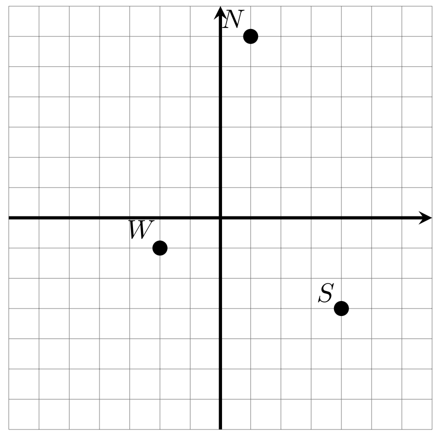


Exploration 2.19. Describe how to draw a taxicab ellipse if you know the foci and the total distance from each point to the two foci.

Exploration 2.20. (a) Taxicab City has two fire stations: Firehouse North located at $(1,6)$ and Firehouse South located at $(4,-3)$. Where should they draw the district boundary line so that all homes within the district are closest to their fire station?



(b) Taxicab City has grown so much, the city council decides to build a third fire station. Firehouse West will be located at $(-2,-1)$. Where should the district boundary lines be now?

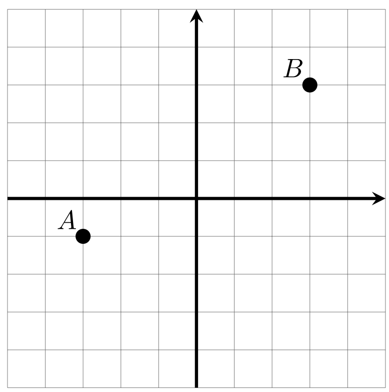


(c) The fire department decides to build a training facility in Taxicab City. Where should they put the training facility if they want it to be equally distant from all three fire stations?

Exercises

Complete your work on separate paper and turn it in. All taxicab pictures should be completed on graph paper. Do not put different problems on the same graph. (For example, problems 4(a) and 4(b) may be done on the same graph, but not problems 4 and 5.)

1. Alice and Bob reside in Taxicab City. Alice works as an administrative assistant at an art school located at $A = (-3, -1)$. Bob works as a bagel baker in a bakery located at $B = (3, 3)$. Alice and Bob just got married and are looking for a house in the city. Alice has always dreamed of a cozy little house on a corner lot, so they will only consider houses located at street corners. Because they will walk to their jobs along the city streets, they measure all their distances using taxicab geometry.



   (a) The newlyweds decide to find a house located so that the number of blocks Alice has to walk to work plus the number of blocks Bob has to walk to work is as small as possible. Where should they look for a house? (Draw the points on a graph.)

   (b) Now Bob decides to be chivalrous and insist that Alice should not have to walk any farther than he does, but they still want the total amount of walking to be minimal. Now where should they look for a house?

   (c) Alice decides to be generous in return, and wants both her husband and herself to walk exactly the same distance to work. They still want the total amount of walking to be minimal. Now where should they look for a house?

   (d) They still haven't found a house! Having decided to widen their search, Alice and Bob keep only the requirement that they both walk the same distance to work. Now where should they look for a house?

   (e) None of the houses they have seen is good enough, so they have a new criterion: they want the total number of blocks they have to walk—Alice plus Bob together—to be at most 12 blocks. Where should they look?

2. Draw a taxicab ellipse with foci $A(1,2)$ and $B(-1,-3)$ so that the total distance from each point on the ellipse to the foci is 10.

3. Draw a taxicab ellipse with foci $A(-2,3)$ and $B(-2,-1)$ so that the total distance from each point on the ellipse to the foci is 8.

4. Every taxicab circle is actually a taxicab square. Is every taxicab square a taxicab circle? Explain your answer.

5. Our taxicab city has two high schools, Taxicab West and Taxicab East. West is at $(-4,3)$ and East is at $(2,1)$.

   (a) Where should they draw the school district boundary lines so that each student attends the high school nearest their home? (As always in our city, we measure distances using taxicab geometry.)

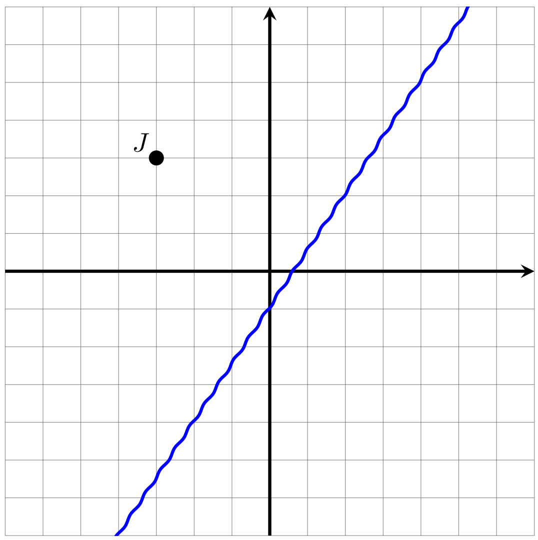
   (b) Our city decides to add a third high school, Taxicab South, at $(-1,-6)$. Now where should we draw the boundary lines?

   (c) If the owner of Pizza Palace wants to set up a pizzeria that is equally distant from all three high schools, where should the owner put the pizzeria?

   (d) Wow! Our city is really growing! If we add a fourth high school, Taxicab North, at $(2,5)$, where should we draw the boundary lines?

2.4 Taxicab Lines

Exploration 2.21. In nearby Omnibus City, a river runs through town on a line running through $(0, -1)$ and $(2, 2)$ as shown.

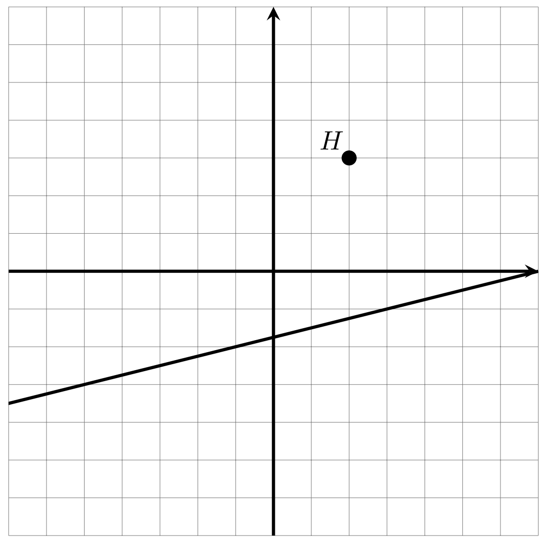


(a) Josephine currently lives in an apartment at $J = (-3, 3)$. What point on the river is closest to her apartment (in taxicab geometry, of course)?

(b) How far is Josephine's apartment from the river (in taxicab geometry, of course)?

(c) Josephine wants to move to a scenic apartment within three blocks' walk of the river. Where should Josephine look for an apartment?

Exploration 2.22. The bike path in Taxicab City runs on a line through $(-5,-3)$ and $(3,-1)$, as shown. Hilda lives at $H = (2,3)$.

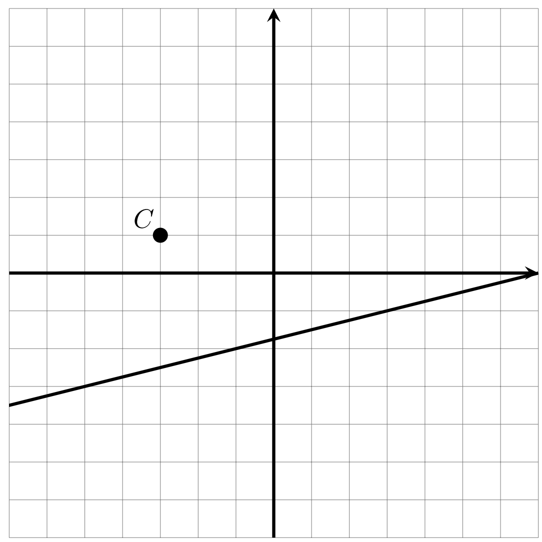


(a) Hilda is recuperating from an accident and can't walk very far. She wants to know where she can go if she only walks two blocks. Draw the taxicab circle representing the places she can visit.

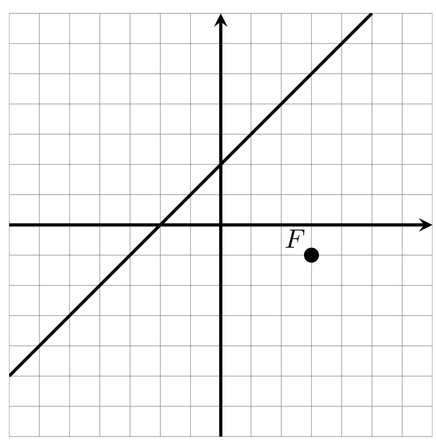
(b) Hilda's recovery is proceeding well from week to week. Draw circles representing how far she can go if she walks 3 blocks, 4 blocks, or 5 blocks.

(c) How far is Hilda's house from the bike path?

Exploration 2.23. The bike path in Taxicab City runs on a line through $(-5,-3)$ and $(3,-1)$, as shown. How far is City Hall $C = (-3, 1)$ from the bike path?



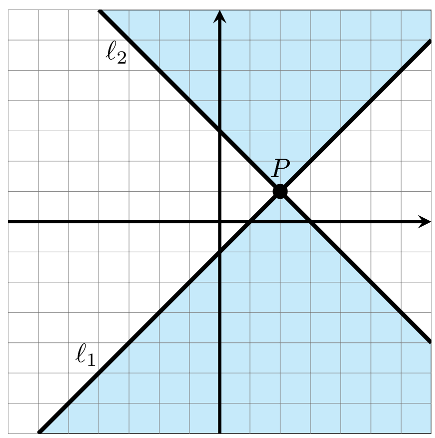
Exploration 2.24. Fred is a city engineer preparing to hook up the electricity to Taxicab City's new football stadium at $F = (3,-1)$. He needs to hook into a main power line that runs along a line from $(-7,-5)$ to $(5,7)$.



His cable needs to be buried along the city streets. How many blocks' worth of cable does he need to reach from the stadium to the power line? What route should the cable take?

Exploration 2.25. The Euclidean method for finding $d_E(A, \ell)$ is often stated as "Measure the distance from A to ℓ along the perpendicular." We would like to formulate a similar easily remembered verbalization for how to find $d_T(A, \ell)$. To do so, it is convenient to introduce some terminology. For any point P in the plane, we classify lines through P under three headings:

- $45°$ lines - ℓ_1 and ℓ_2 are the only $45°$ through P.
- Steep lines - Any line through P that lies in the shaded region below.
- Shallow lines - Any line through P that lies in the unshaded region.



Complete these statements that tell how to determine the taxicab distance from A to ℓ

(a) "If ℓ is steep, measure the distance from A to L along ____________________

____."

(b) "If ℓ is shallow, measure the distance from A to L along ____________________

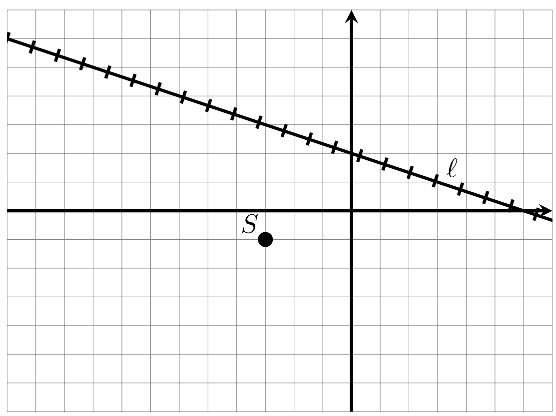
____."

(c) "If ℓ is a $45°$ line, measure the distance from A to L along ____________________

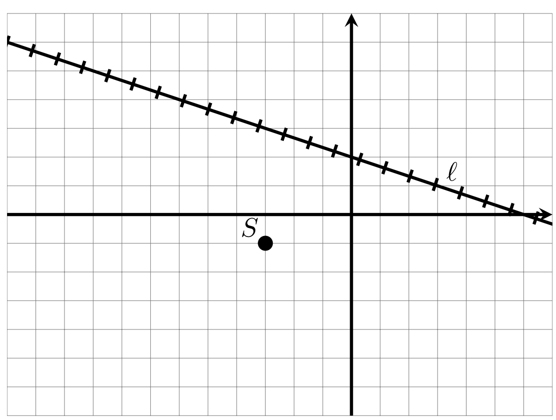
____."

Exploration 2.26. Susan and Leo are moving to town. Susan got a job at Sushi Hut at $S = (-3, -1)$, whereas Leo will be working for the city light rail line ℓ that runs through the city as shown. One of Leo's fringe benefits is that when he comes to work he can just get on the train wherever is closest to his home.

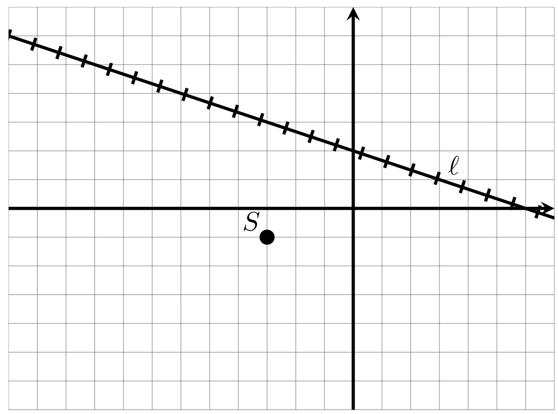
(a) Susan and Leo want to live where the distance Susan has to walk to work plus the distance Leo has to walk to work is a minimum. Where should they look?



(b) They change their minds and decide to live where they both walk the same distance to work. Where should they look?



(c) Where should they look if all that matters is that Susan have a shorter distance to walk than Leo?



Exercises

Complete your work on separate paper and turn it in. All taxicab pictures should be completed on graph paper. Do not put different problems on the same graph. (For example, problems 4(a) and 4(b) may be done on the same graph, but not problems 4 and 5.)

1. Find the taxicab distance between the point $A = (1, 2)$ and the line that passes through the points $(3, 0)$ and $(4, 2)$.

2. (a) Draw the line that passes through $(0, 3)$ and $(-4, 1)$. Call it ℓ.

   (b) Find two different points that are a taxicab distance of 2 from the line ℓ. Name their coordinates.

   (c) Draw a picture of *all* the points that are a taxicab distance of 2 from the line ℓ. (Points with non-integer coordinates are allowed!)

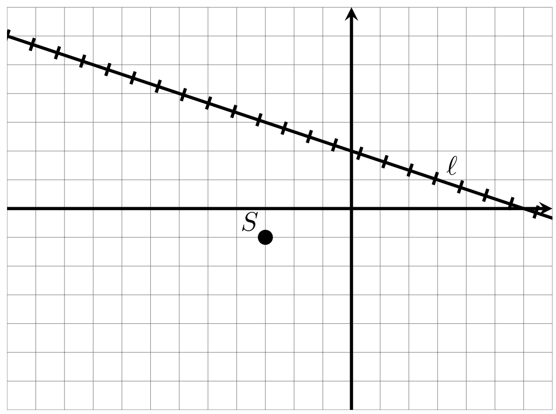
3. (a) Draw the line that passes through $(0, 0)$ and $(1, 3)$. Call it ℓ.

   (b) Find two different points that are a taxicab distance of 4 from the line ℓ. Name their coordinates.

   (c) Draw a picture of *all* the points that are a taxicab distance of 4 from the line ℓ. (Points with non-integer coordinates are allowed!)

4. Find a line ℓ carefully chosen so that the collection of points that are a taxicab distance of 3 from ℓ and the collection of points that are a Euclidean distance of 3 from ℓ are the same. In other words, every point that is a taxicab distance of 3 from ℓ is also a Euclidean distance of 3 from ℓ, and vice versa.

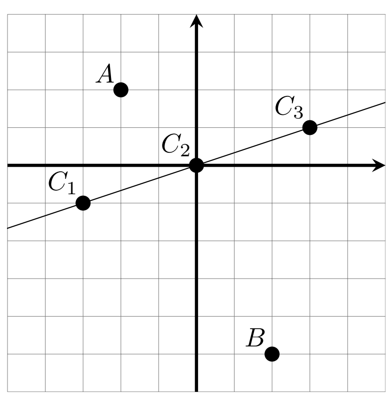
5. Susan got a job at Sushi Hut at $S = (-3, -1)$, whereas Leo will be working for the city light rail line ℓ that runs through the city as shown. One of Leo's fringe benefits is that when he comes to work he can just get on the train wherever is closest to his home.



   (a) Susan and Leo want to live where the distance Susan has to walk to work is no more than 2 blocks and the distance Leo has to walk to work is no more than 3 blocks. Where should they look?

   (b) They change their minds and decide to live where they both walk three blocks to work. Where should they look?

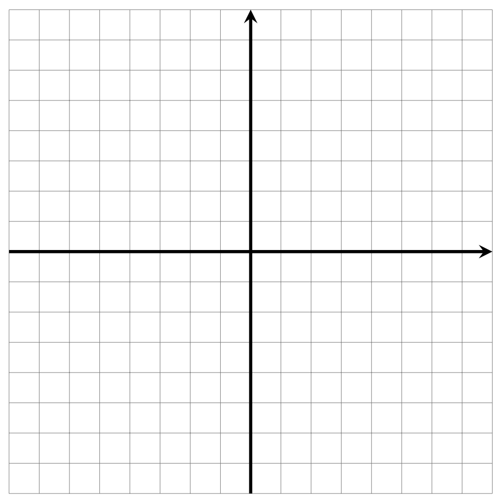
6. Alex lives at the point $A = (-2, 2)$ in Taxicab City, and Bonnie lives at the point $B = (2, -5)$. A railroad track has been constructed along the line joining $(0, 0)$ and $(3, 1)$ as shown, and there are only three places to cross the railroad. The crossings are at $C_1 = (-3, -1)$, $C_2 = (0, 0)$, and $C_3 = (3, 1)$. If Alex wants to walk to Bonnie's house using one of the three crossings, which crossing should he use, and how many blocks does it take him in total to get to Bonnie's house?



2.5 Taxicab Triangles

The Pythagorean Theorem in Euclidean geometry states that in a right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse (the side opposite the right angle).

Exploration 2.27. (a) Draw three right triangles on the graph.



(b) Measure their sides using taxicab distances.

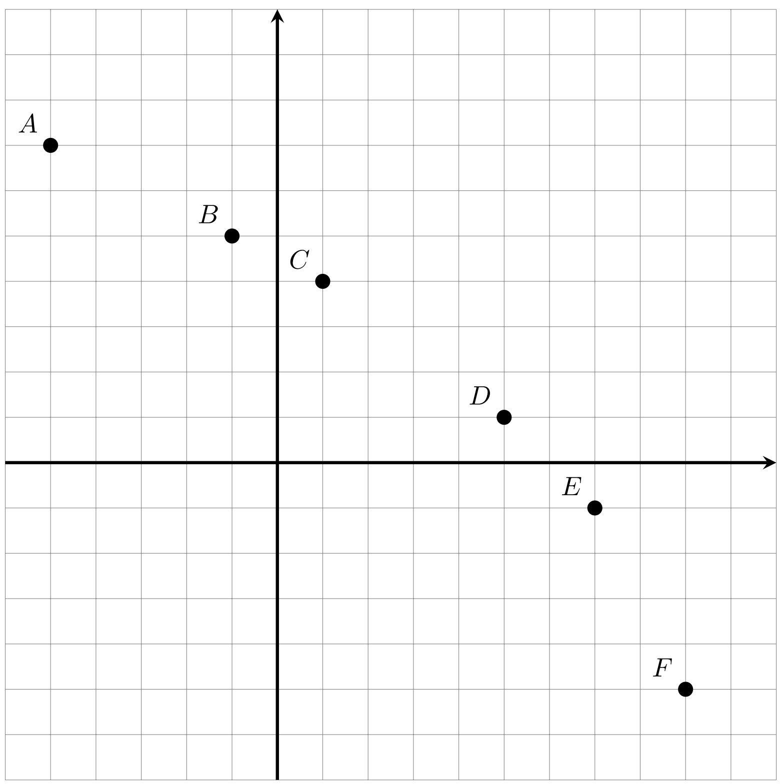
	Leg (short side)	Leg (short side)	Hypotenuse (long side)
Triangle I			
Triangle II			
Triangle III			

(c) Does the Pythagorean Theorem work in taxicab geometry? Why or why not?

Exploration 2.28. Can you come up with a replacement for the Pythagorean Theorem in taxicab geometry? In other words, if you have a right triangle with legs of length a and b, can you find a formula for the length of the hypotenuse c?

The midset of two points is the set of all points equidistant from the two points. Note that in Euclidean geometry, the midset of two points is the perpendicular bisector of the line segment connecting the two points.

Exploration 2.29. Consider the points in the following diagram.



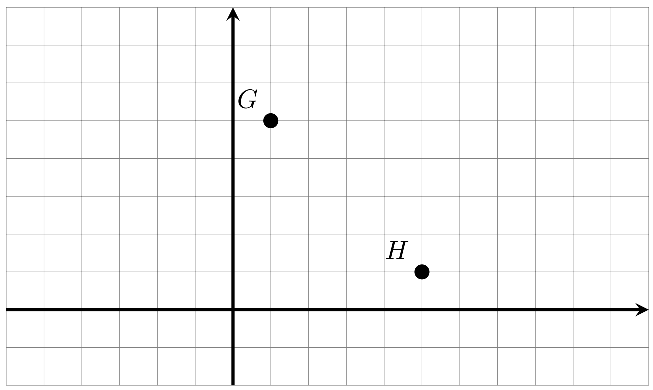
(a) Find all points that are the same distance from A as from B.

(b) Find all points that are the same distance from C as from D.

(c) Find all points that are the same distance from E as from F.

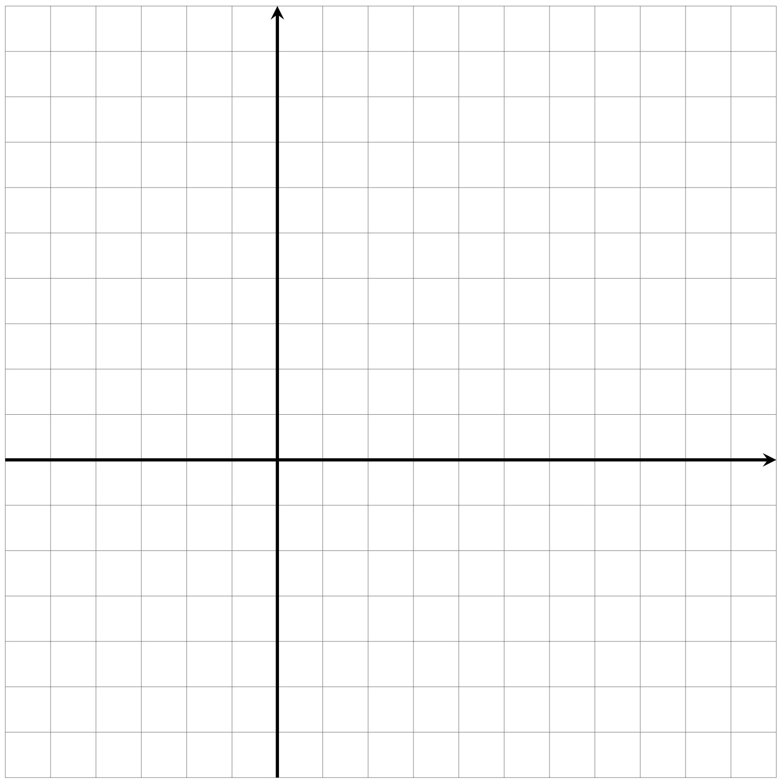
Exploration 2.30. Given two points X and Y, describe a rule for finding the midset of X and Y.

Exploration 2.31. Find the midset of G and H.



Exploration 2.32. When, if ever, is the taxicab midset the same as the Euclidean midset?

Exploration 2.33. Let $A = (6, -2)$, $B = (0, 6)$, and $C = (8, 4)$.



(a) Draw the triangle $\triangle ABC$.

(b) Find the midset of A and B.

(c) Find the midset of A and C.

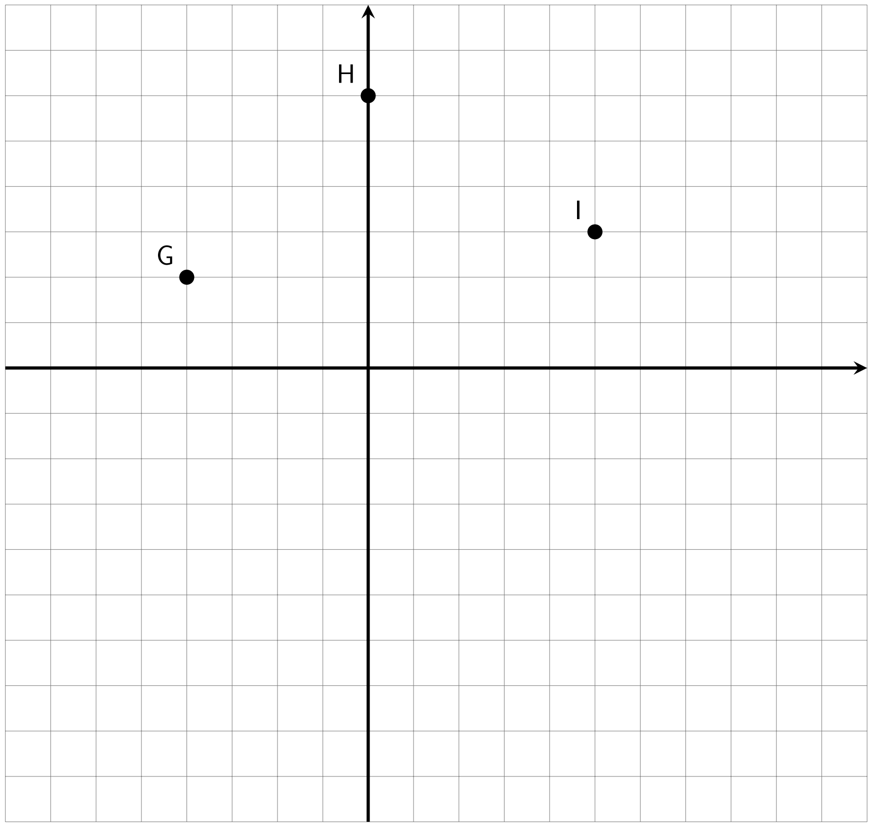
(d) Find the midset of B and C.

(e) Let P be the point where those three sets meet. What are the coordinates of P? How far is P from each of the three points A, B, and C?

(f) Draw a taxicab-circle that touches each of the three corners of $\triangle ABC$.

The circumscribed circle or circumcircle of a triangle is a circle which passes through all three vertices of the triangle. The center of this circle is called the circumcenter.

Exploration 2.34. Consider triangle $\triangle GHI$, where $G = (-4, 2)$, $H = (0, 6)$, and $I = (5, 3)$.

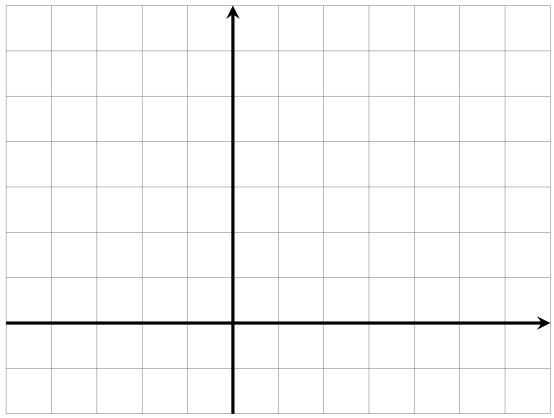


(a) Circumscribe a taxicab-circle around triangle $\triangle GHI$.

(b) Circumscribe a *different* taxicab-circle around $\triangle GHI$.

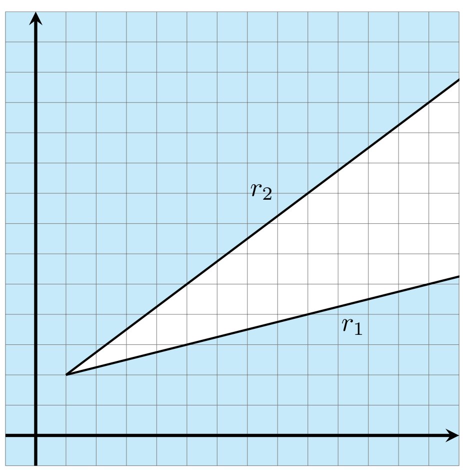
(c) How many different taxicab-circles *can* be circumscribed around $\triangle GHI$?

Exploration 2.35. Circumscribe a taxicab-circle around triangle $\triangle DEF$, where $D = (-3, 0)$, $E = (0, 1)$, and $F = (5, 5)$.



The midset of two lines is the set of all points equidistant from the two lines. Note that in Euclidean geometry, the midset of two intersecting lines is a pair of lines that form angle bisectors for the original lines.

Exploration 2.36. Consider the two rays (half-lines) r_1 and r_2 that make an angle in the following diagram.



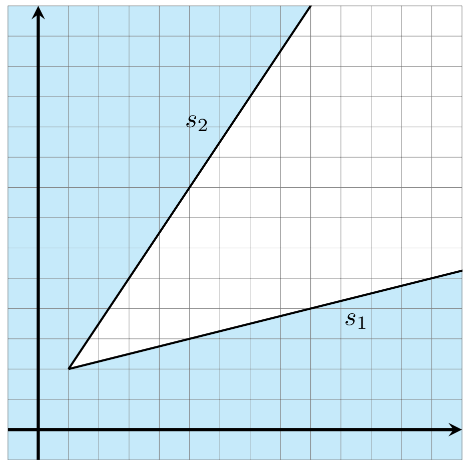
(a) Is r_1 steep or shallow? Is r_2 steep or shallow?

(b) Find a point A within the angle (unshaded region) such that the distance from A to r_1 is the same as the distance from A to r_2.

(c) Find two more such points.

(d) Connect the dots to find *all* points P in the angle such that the distance from P to r_1 is the same as the distance from P to r_2.

Exploration 2.37. Consider the two rays (half-lines) s_1 and s_2 that make an angle in the following diagram.



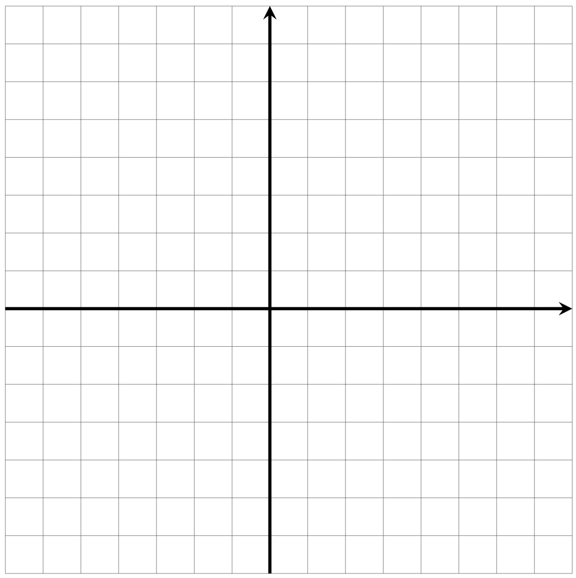
(a) Is s_1 steep or shallow? Is s_2 steep or shallow?

(b) Find a point B within the angle (unshaded region) such that the distance from B to s_1 is the same as the distance from B to s_2.

(c) Find two more such points.

(d) Connect the dots to find *all* points P in the angle such that the distance from P to s_1 is the same as the distance from P to s_2.

Exploration 2.38. Let $A = (-2, -5)$, $B = (-2, 4)$, and $C = (4, 7)$.



(a) Draw the triangle $\triangle ABC$.

(b) Draw the line containing all points equally distant from side AB and side BC.

(c) Draw the line containing all points equally distant from side AB and side AC.

(d) Draw the line containing all points equally distant from side BC and side AC.

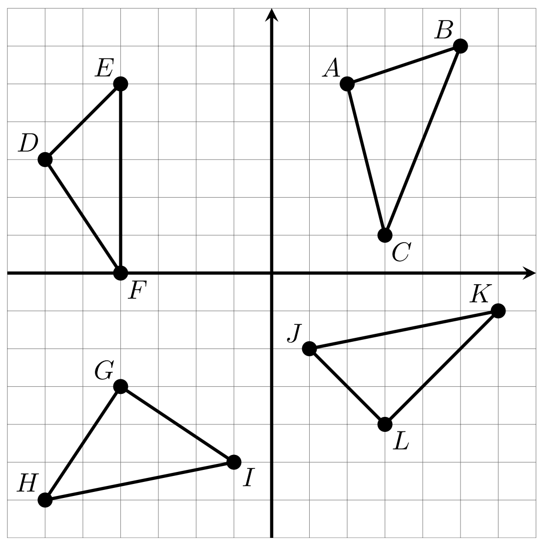
(e) Let P be the point where the three lines meet. What are the coordinates of P? How far is P from each of the three sides?

(f) Draw a taxicab-circle that touches each of the three sides of $\triangle ABC$.

Exercises

Complete your work on separate paper and turn it in. All taxicab pictures should be completed on graph paper. Do not put different problems on the same graph. (For example, problems 4(a) and 4(b) may be done on the same graph, but not problems 4 and 5.)

1. (a) Draw a triangle that has side lengths of 3, 4, and 5 *in Euclidean distance*.
   (b) Draw a triangle that has side lengths of 3, 4, and 5 *in taxicab distance*.
2. Of triangles $\triangle ABC$, $\triangle DEF$, $\triangle GHI$, and $\triangle JKL$, which are taxicab-isosceles? (Recall that an isosceles triangle is one that has at least two sides of equal length.)



3. Draw a taxicab-equilateral triangle which has the line segment from $(1,1)$ to $(7,1)$ as one side. (Recall that an equilateral triangle is one in which all three sides have equal length.)
4. Draw a taxicab-equilateral triangle, none of whose sides is vertical and none of whose sides is horizontal.
5. Alice and Bob have adopted a boy named Charles, who attends Chesterton Elementary School at $C = (0, -3)$. Alice still works at at the art school $A = (-3, -1)$ and Bob at the bagel bakery at $B = (3, 3)$.
   (a) Where should they live so that each of the three of them has the same distance to walk to work or school?
   (b) Where should they live if they have decided that Charles should have the shortest walk, Alice the second shortest walk, and Bob the longest walk? Shade the appropriate region of your graph.

3

Probability

Probability is apparent in all of our everyday lives – people buy lottery tickets, purchase car insurance, and participate in games of chance. In this chapter we will be looking at many of the important concepts of probability in our lives. We will begin with some of the basics and simple experiments and continue on to look at different methods of computing probability and computing odds.

3.1 Probability Basics and Simple Experiments

An **experiment** is a process that can be repeated and may produce different **outcomes**, or results. The set of all possible outcomes is the **sample space**. We say that the outcomes are **equally likely** if each outcome of an experiment has the same chance of occurring as any other outcome. An **event** is some subset of the possible outcomes of an experiment.

Example 3.1. List the sample space and one possible event for each of the following experiments.

(a) *Experiment:* Toss a coin 2 times and record the results in order.

(b) *Experiment:* Roll two standard dice and record the the results in order.

Solution. (a) *Sample Space:* We will use two-letter sequences to represent the possible outcomes. Below we list all possible outcomes with H denoting heads and T denoting tails.

HH HT TH TT

Thus, there are 4 possible outcomes in this sample space. The sample space is given by the set $\{$HH, HT, TH, TT$\}$.

Event: One possible event is the subset $\{\text{HH,HT}\}$, which is the event of getting heads on the first coin. There are many other possible subsets.

(b) *Sample Space:* We will use ordered pairs to represent the possible outcomes. The first number represents the number appearing on the first die and the second number represents the number appearing on the second die. The sample space consists of $6 \times 6 = 36$ outcomes which are listed below.

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array}$$

Event: Rolling the die so that the total on the dice is 6 is one possible event. This event has a total of 5 outcomes: $\{(5,1),(4,2),(3,3),(2,4),(1,5)\}$. Again, there are many other possible events. ■

Experimental Probability

The probability of an event can be found by carrying out a series of experiments. For example, if we wanted to find the probability of rolling a 3 on one six-sided die, we could roll the die repeatedly. We would record the number of times we rolled a 3 and divide it by the total number of times we rolled the die. This would lead us to find the *experimental probability* of the event. **Experimental probability**, sometimes called **empirical probability**, is the relative frequency with which an event occurs in a particular sequence of trials. Note that, since the probability is based on actual observations of a particular experiment, experimental probability can differ from one situation to the next. An experimental probability of 1, or 100%, indicates that the event has always occurred.

Example 3.2. An experiment requires two coins to be flipped 300 times. The results were recorded in the following table. If E is the event of getting a tail on the first coin, find the experimental probability of E.

Outcome	Frequency	Experimental Probability
HH	72	$\frac{72}{300}$
HT	98	$\frac{98}{300}$
TH	86	$\frac{86}{300}$
TT	44	$\frac{44}{300}$
	Total: 300	Total: $\frac{300}{300} = 1$

Solution. The event E is $\{$TH,TT$\}$. We can see that a tail appeared on the first coin $86+44=130$ times. Therefore, the experimental probability of E is

$$\frac{130}{300}=0.4\bar{3}$$

■

Theoretical Probability

Experimental probability is convenient because it can be calculated by simply performing a series of experiments; however, it is inconvenient since it may produce a different result every time. Thus, we use *theoretical probabilities* to determine how likely it is that an event will occur without actually performing the experiments. For example, you probably know that a coin is going to land heads about $\frac{1}{2}$ of the time and that a die is going to land showing a 4 about $\frac{1}{6}$ of the time. These are both theoretical probabilities because they are what *should* happen in a perfect world.

Suppose that all the outcomes in the sample space S are equally likely to occur. Let E be an event. The **theoretical probability** of event E, $P(E)$, is

$$P(E)=\frac{\text{number of outcomes in } E}{\text{number of outcomes in } S}$$

For example, rolling one six-sided die is an experiment that may produce six different outcomes: rolling a $1,\ 2,\ 3,\ 4,\ 5,$ or 6. (Unless otherwise stated, we will assume the die we use are "fair" die, that is each number on the die has an equal chance of begin rolled.) Rolling an odd number is an example of an event; this event occurs when you roll a $1,\ 3,$ or 5.

Since we know that the number of outcomes that result in event E must be less than or equal to the total number of outcomes, we can say that $0\le P(E)\le 1$. The probability of an event can be reported as a fraction, a decimal, or a percentage. (If represented as a percentage, we say $0\%\le P(E)\le 100\%$.) The greater the probability, the more likely the event is to occur.

Example 3.3. A die is rolled. Find the probability of rolling a(n)

(a) 4.

(b) 7.

(c) odd number.

(d) number greater than 1.

(e) number less than 7.

Solution. (a) There are six equally likely outcomes: $1,\ 2,\ 3,\ 4,\ 5,$ and 6. The event of rolling a 4 can only occur one way.

$$P(4) = \frac{\text{number of outcomes that will result in a 4}}{\text{total number of possible outcomes}} = \frac{1}{6}$$

(b) There are no outcomes that will result in a 7.

$$P(7) = \frac{0}{6} = 0$$

Thus the probability is 0.

(c) The event of rolling an odd number can occur in 3 ways: $1,\ 3,$ or 5.

$$\begin{aligned} P(\text{odd number}) &= \frac{\text{number of outcomes that result in an odd number}}{\text{total number of possible outcomes}} \\ &= \frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$

(d) Five numbers are greater than 1: $2,\ 3,\ 4,\ 5,$ and 6.

$$P(\text{number greater than 1}) = \frac{5}{6}$$

(e) All of the outcomes 1 through 6 are less than 7.

$$P(\text{number less than 7}) = \frac{6}{6} = 1$$

Thus the probability is 1. ■

Probability Facts to Note:

- The probability of an event that cannot occur is 0.
- The probability of an event that must occur is 1.
- For any event E, $0 \leq P(E) \leq 1$.
- The sum of the probabilities of all possible outcomes of an experiment is 1.

Example 3.4. Three cards are drawn at random from a standard 52-card deck without replacing them. What is the probability that

(a) all three cards are hearts?

(b) all three cards are face cards (jack, queen, or king)?

(c) all three cards are black?

Solution. (a) In a standard deck of cards, there are 13 hearts. The number of ways to have a three card hand that is all hearts is ${}_{13}C_3 = \dfrac{13!}{10!3!} = 286$ and the number of ways to have any three card hand is ${}_{52}C_3 = \dfrac{52!}{49!3!} = 22,100$. So the probability of drawing three hearts is

$$P(3\ \heartsuit) = \frac{286}{22,100} = \frac{11}{850} \approx 0.0129$$

(b) In a standard deck of cards, there are 12 face cards. The number of ways to have a three card hand that is all face cards is ${}_{12}C_3 = \dfrac{12!}{9!3!} = 220$ and the number of ways to have any three card hand is ${}_{52}C_3 = \dfrac{52!}{49!3!} = 22,100$. So the probability of drawing three face cards is

$$P(\text{face card}) = \frac{220}{22,100} = \frac{11}{1105} \approx 0.0100$$

(c) In a standard deck of cards, there are 26 black cards. The number of ways to have a three card hand that is all black cards is ${}_{26}C_3 = \dfrac{26!}{23!3!} = 2600$ and the number of ways to have any three card hand is ${}_{52}C_3 = \dfrac{52!}{49!3!} = 22,100$. So the probability of drawing three black cards is

$$P(\text{black}) = \frac{2600}{22,100} = \frac{2}{17} \approx 0.1176$$

■

Example 3.5. We toss two dice. Let A be the event of rolling a total of 3, B be the event of rolling a total of 9, and C be the event of rolling at least at total of 6. Find the theoretical probabilities $P(A)$, $P(B)$, and $P(C)$.

Solution. We found earlier that rolling 2 dice has a total of 36 possible outcomes. Looking at the list of possible outcomes when rolling two dice, we see that there are 2 ways to roll a total of 3 – $A = \{(1,2),(2,1)\}$. Similarly, there are 4 outcomes in B and 26 outcomes in C. The theoretical probabilities of A, B, and C are shown in the following table.

Event	Number of Outcomes	Probability
A	2	$P(A) = \frac{2}{36} = \frac{1}{18}$
B	4	$P(B) = \frac{4}{36} = \frac{1}{9}$
C	26	$P(C) = \frac{26}{36} = \frac{13}{18}$

■

Law of Large Numbers

In situations where we can determine the theoretical probability of an event, how does it compare to the experimental probability of the event? Most of us would accept that if a "fair" coin is tossed many times, it will land heads up approximately half of the time. Note that the theoretical probability of the coin landing heads up is one half. Does this mean that if the coin is tossed 4 times it will come up heads exactly twice? If we tossed the coin 20 times it will come up heads exactly 10 times? The answer is clearly no. Consider the following data collected from tossing a "fair" coin.

Number of Tosses	Expected Number of Heads	Actual Number of Heads Observed	Experimental Probability
10	5	4	$\frac{4}{10} = 0.4$
100	50	52	$\frac{52}{100} = 0.52$
1000	500	472	$\frac{472}{1000} = 0.472$
10,000	5000	5102	$\frac{5102}{10,000} = 0.5102$
100,000	50,000	49,897	$\frac{49,897}{100,000} = 0.49897$

Note that as the number of tosses increases, the experimental probability gets closer and closer to 0.5, which is the theoretical probability. This property is known as the Law of Large Numbers – (theoretical) probability statements apply in practice to a large number of trials, not to a single trial. It is the relative frequency over the long run that is predictable, not individual events or precise totals.

Exercises

Calculating probabilities requires a lot of work with fractions. In Exercises 1 - 3, simplify the expression as much as possible.

1. (a) $\dfrac{3}{8}+\dfrac{3}{4}$ (b) $\dfrac{5}{8}-1+\dfrac{5}{6}$

   (c) $\dfrac{\frac{6}{7}}{\frac{5}{2}}$ (d) $\dfrac{3}{2}\times\dfrac{5}{7}$

2. (a) $\dfrac{\frac{9}{11}}{\frac{5}{7}}$ (b) $\dfrac{3}{2}\times\dfrac{1}{4}+\dfrac{7}{2}\times\dfrac{9}{2}$

   (c) $\dfrac{1}{3}\times 120+\dfrac{2}{5}\times(-60)$ (d) $\dfrac{\frac{4}{7}}{1-\frac{4}{7}}$

3. (a) $\dfrac{2}{3}+12-\dfrac{14}{3}$ (b) $\dfrac{1}{5}\times 500+\dfrac{10}{3}-\dfrac{3}{4}\times 200$

   (c) $\dfrac{\frac{5}{4}}{\frac{4}{7}-1}$ (d) $\dfrac{6}{7}+\dfrac{4}{9}$

4. List the outcomes in the sample space and one possible event for each of the following experiments.
   (a) A coin is flipped three times and the results are recorded.
   (b) Two fair six-sided dice are rolled and the number on the face of each is recorded.
   (c) A customer takes a pen out of the jar containing both blue and black pens. The color of the pen is recorded.

5. An experiment consists of flipping a coin three times.
   (a) List all possible outcomes.
   (b) What is the probability of getting three heads?
   (c) What is the probability of getting no heads?
   (d) What is the probability of getting exactly two heads?

6. A standard six-sided die is rolled 100 times with the following results.

Outcome	Frequency
1	32
2	11
3	8
4	13
5	10
6	26

   Find the experimental and theoretical probabilities of

   (a) rolling a 6.
   (b) rolling an even number.
   (c) rolling a number less than 4.

7. A single six-sided die is rolled. Find the probability of rolling a(n)

   (a) 2.
   (b) number less than 3.
   (c) even number.
   (d) 8.

8. One card is drawn at random from a standard 52-card deck. What is the probability that the card will be a

   (a) diamond?
   (b) king?
   (c) heart or a spade?
   (d) face card?

9. A multiple choice question has 5 possible answers, but only one is correct. If you guess because you do not know the answer, what is the probability that you will guess the correct answer?

10. Two six-sided dice are rolled. What is the probability that the sum of the two dice will be 8?

11. A class consists of 19 girls and 15 boys. If 12 of the students are to be selected at random, determine the probability that they are all girls.

12. Three letters are to be selected at random from the English alphabet of 26 letters. Determine the probability that 3 vowels (a, e, i, o, u) are selected.

13. A class of 16 people contains 4 people whose birthday is in October. If 3 people from the class are selected at random, determine the probability that none of those selected has an October birthday.

For Exercises 14 and 15, an experiment consists of rolling two fair six-sided dice. Give the theoretical probabilities of the following events.

14. (a) Rolling a 6 on the first die
    (b) Rolling an odd number on the second die
    (c) Rolling a total of 8
    (d) Rolling a total of 16

15. (a) Rolling a total greater than 1
    (b) Rolling an odd number on the first die and an even number on the second die
    (c) Rolling a 3 on the second die
    (d) Rolling a total of less than 8

In Exercises 16-19, a television game show has five doors, of which the contestant must pick two. Behind two of the doors are expensive cars, and behind the other three doors are consolation prizes. The contestant gets to keep the items behind the two doors she selects. Determine the probability that the contestant wins

16. no cars.

17. both cars.

18. at least one car.

19. exactly one car.

3.2 Probability of Events Involving "Not" and "Or"

In this section, we will begin by looking at **compound probability** problems that contain the words *not* or *or* without having to construct a sample space.

Complement of an Event

Sometimes when calculating the probability of an event occurring, we also want to know the probability that an event will *not* occur. The **complement of the event** E is the set of outcomes in a sample space S, but not in an event E. We write the complement of E as $\bar{E}$ (read "not E").

The probability of an event occurring is related to the probability of the complement occurring in the following way

$$P(\bar{E}) + P(E) = 1$$

Therefore, if we know either $P(E)$ or $P(\bar{E})$, we are able to find the other.

Example 3.6. Julie and Jenni are playing a game and are about to roll a six-sided die to see who gets to go first. Assume that it is a fair die and all outcomes are equally likely.

(a) What is the probability that Julie and Jenni will roll the same number?

(b) What is the probability that they roll different numbers?

Solution. We will first list the 36 outcomes as ordered pairs of numbers from 1 to 6, the first being Julie's rolled number, the second Jenni's rolled number.

(1,1)	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$
$(2,1)$	**(2,2)**	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$
$(3,1)$	$(3,2)$	**(3,3)**	$(3,4)$	$(3,5)$	$(3,6)$
$(4,1)$	$(4,2)$	$(4,3)$	**(4,4)**	$(4,5)$	$(4,6)$
$(5,1)$	$(5,2)$	$(5,3)$	$(5,4)$	**(5,5)**	$(5,6)$
$(6,1)$	$(6,2)$	$(6,3)$	$(6,4)$	$(6,5)$	**(6,6)**

(a) Let E be the event that the numbers rolled by Julie and Jenni are the same. Outcomes for E are shown in boldface. Thus, we have

$$P(E) = \frac{6}{36} = \frac{1}{6}$$

(b) If E is the event that the numbers rolled are the same, then $\bar{E}$ is the event that the numbers rolled are different. Thus, the probability that the numbers are different is

$$P(\bar{E}) = 1 - P(E) = 1 - \frac{1}{6} = \frac{5}{6}$$

We can also calculate this probability directly by counting the outcomes not in E and finding that there are 30. Thus,

$$P(\bar{E}) = \frac{30}{36} = \frac{5}{6}$$

■

Or Problems

The **union** of two events $A \cup B$ refers to all outcomes that are in A, in B, or in both. The **intersection** of two events $A \cap B$ refers to all outcomes that are in both A and B. Figure 3.1 shows a Venn diagram that illustrates union and intersection. Thus, the union of two events, $A \cup B$, relates to A *or* B happening, while the intersection of two events $A \cap B$, relates to A *and* B happening.

An **or probability problem** requires a successful outcome for at least one of the given events. For example, consider a situation in which we roll a die and want to find the probability of rolling a 1 *or* a 2. In this situation, rolling a 1 or rolling a 2 would both be considered successful. The *addition rule of probability* is used to find the probability of event A or event B, represented as $P(A \cup B)$. The **addition**

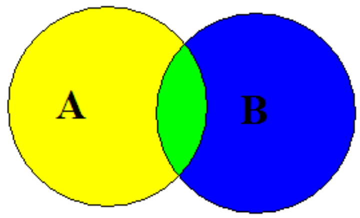


Figure 3.1: $A \cup B$ is the yellow, blue and green regions. $A \cap B$ is the green region.

rule of probability states that the probability of event A or B or both occuring is equal to the sum of the probabilities of events A and B less the probability that both events occur. In symbols,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example 3.7. One card is selected at random from a standard 52-card deck. Find $P(A \cup B)$ for the given events.

(a) $A =$ a face card, $B =$ a black card

(b) $A =$ an ace, $B =$ a diamond

(c) $A =$ a red card, $B =$ a black card

Solution. (a) There are 12 face cards in the deck, there are 26 black cards in the deck, and 6 of the 12 face cards are black. Thus,

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{12}{52} + \frac{26}{52} - \frac{6}{52} \\ &= \frac{32}{52} \\ &= \frac{8}{13} \end{aligned}$$

(b) There are 4 aces in the deck, 13 diamonds in the deck, and one card is both an ace and a diamond. Thus,

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} \\ &= \frac{16}{52} \\ &= \frac{4}{13} \end{aligned}$$

(c) There are 26 red cards and 26 black cards in the deck. It is impossible to select a card that is both a red card and a black card, so $P(A \cap B) = 0$. Thus,

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{26}{52} + \frac{26}{52} - 0 \\ &= 1 \end{aligned}$$

Since $P(\text{red or black}) = 1$, a red card or a black card must be selected. ■

Example 3.8. A card is drawn at random from a standard 52-card deck. Let A be the event the card is a diamond and let B be the event the card is a face card. Find and interpret $P(A \cup B)$.

Solution. 52 equally likely outcomes make up the sample space. Event A has 13 outcomes because there are 13 diamonds in a deck, so $P(A) = \frac{13}{52}$. Event B has 12 outcomes, so $P(B) = \frac{12}{52}$. There are three cards that are both face cards and diamonds. Thus, $P(A \cap B) = \frac{3}{52}$. To find $P(A \cup B)$, we use the given formula.

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\ &= \frac{22}{52} \\ &= \frac{11}{26} \end{aligned}$$

$P(A \cup B) = \frac{11}{26}$ is the probability that the drawn card is a face card, a diamond, or both. ■

Computing Odds

Many times, the chances of an event occurring are expressed in terms of odds rather than probability. When discussing the odds of winning the lottery, the odds given are always "odds against", unless specified otherwise. The **odds against** an event is a ratio of the probability the event will fail to occur to the probability the event will occur. The **odds in favor** of an event is a ratio of the probability the event will occur to the probability the event will fail to occur. Thus, we have

$$\text{Odds against an event } E \;-\; P(\bar{E}) : P(E)$$

$$\text{Odds in favor of an event } E \;-\; P(E) : P(\bar{E})$$

Example 3.9. If event E has a probability of $\frac{2}{5}$, find (a) the odds in favor or E, and (b) the odds against E.

Solution. (a)

$$\frac{\frac{2}{5}}{1-\frac{2}{5}} = \frac{\frac{2}{5}}{\frac{3}{5}} = \frac{2}{5} \times \frac{5}{3} = \frac{10}{15} = \frac{2}{3}$$

So the odds in favor of E are $2:3$.

(b)

$$\frac{1-\frac{2}{5}}{\frac{2}{5}} = \frac{\frac{3}{5}}{\frac{2}{5}} = \frac{3}{5} \times \frac{5}{2} = \frac{15}{10} = \frac{3}{2}$$

So the odds against E are $3:2$. ■

Given the odds in favor of an event, we are also able to compute the probability of an event occurring. If we are given that the odds in favor of an event E are $a:b$, then there are a outcomes in E and there are b outcomes in $\bar{E}$. Thus, there are $a+b$ outcomes in the sample space. Hence, we have

$$P(E) = \frac{a}{a+b}$$

when given that the odds in favor of an event E are $a:b$.

Example 3.10. Find $P(E)$ when given the following odds.

(a) The odds in favor of E are $3:4$.

(b) The odds against E are $4:7$.

Solution. (a) $P(E) = \frac{3}{3+4} = \frac{3}{7}$.

(b) If the odds against E are $4:7$, then the odds in favor of E are $7:4$.

$$P(E) = \frac{7}{7+4} = \frac{7}{11}$$

■

Exercises

1. If the probability that an event occurs is 0.9, what is the probability that the event does not occur?

2. An 8-sided die is constructed that has two faces marked with 2s, two faces marked with 3s, two faces marked with 5s, and two faces marked with 8s. If this die is rolled a single time, find the probability of

   (a) Getting a 2.

   (b) Not getting a 2.

   (c) Getting an even number.

   (d) Not getting an even number.

3. A 16-sided die is constructed that has three faces marked with 7s, five faces marked with 1s, four faces marked with 5s, and four faces marked with 4s. If this die is rolled a single time, find the probability of

   (a) Getting a 1.

   (b) Not getting a 1.

   (c) Getting an even number.

   (d) Not getting an even number.

4. As the winner of a raffle, you get to choose between 3 prizes hidden behind curtains 1, 2, and 3. One prize is a brand new sports car and the other two are identical bicycles.

   (a) List all possible arrangements of the prizes behind the curtains.

   (b) Let D be the event that you find the car behind curtain 3. List the outcome(s) of the sample space that correspond to D.

   (c) Describe $\bar{D}$ and list the outcome(s) of the sample space that correspond to $\bar{D}$.

   (d) Find $P(D)$ and $P(\bar{D})$.

In Exercises 5 - 8, find the indicated probability.

5. If $P(A) = 0.4$, $P(B) = 0.7$, and $P(A \cap B) = 0.2$, find $P(A \cup B)$.

6. If $P(A \cup B) = 0.8$, $P(A) = 0.6$, and $P(B) = 0.4$, find $P(A \cap B)$.

7. If $P(A \cup B) = 0.9$, $P(A) = 0.6$, and $P(A \cap B) = 0.3$, find $P(B)$.

8. If $P(A \cup B) = 0.7$, $P(B) = 0.2$, and $P(A \cap B) = 0.1$, find $P(A)$.

In Exercises 9 - 12, one card is drawn at random from a standard 52-card deck. Assuming aces are high, find the probability of selecting

9. an ace or a king.
10. a card less than an 8.
11. a face card or a black card.
12. a card greater than 9 or a red card.
13. 12% percent of the world's population have type B blood, 15% are Rh-negative, and 2% have type B blood and are Rh-negative. What is the probability that a randomly selected individual will have type B blood or be Rh-negative?
14. According to the U.S. Bureau of the Census, the total population of the United States in 1990 was 248,709,873. Of this population, 18,354,443 were under 5 years old and 13,135,272 were over 74 years old. What is the probability that a randomly selected person in the United States in 1990 was over 74 or under 5?
15. According to the American Medical Association, in 1996 there were 737,764 physicians in the United States, 157,387 of whom were female. There were 133,005 physicians under 35 years of age, 47,348 of whom were female. What is the probability that a randomly chosen physician in 1996 was female or under the age of 35?
16. If the probability that an event *will* occur is $\frac{2}{3}$, find
    (a) the probability that the event will not occur.
    (b) the odds against the event occurring.
    (c) the odds in favor of the event occurring.
17. A six-sided die is tossed. Find the odds against and in favor of rolling a(n)
    (a) 3.
    (b) even number.
    (c) number greater than 4.
    (d) number less than 4.
18. A card is drawn at random from a standard 52-card deck. Assuming aces are high, find the odds against and in favor of drawing a(n)
    (a) face card.
    (b) club.
    (c) ace.
    (d) card less than 5.
19. Find $P(E)$ given the following odds.
    (a) The odds in favor of E are $2:3$.
    (b) The odds against E are $9:14$.

20. Find $P(E)$ given the following odds.
    (a) The odds in favor of E are $5:8$.
    (b) The odds against E are $1:2$.
21. Find the odds in favor of event E, when E has the following probabilities
    (a) $\frac{5}{6}$ (b) $\frac{1}{8}$

3.3 Conditional Probability and Events Involving "And"

Conditional Probability

Suppose two events A and B are in a sample space S and that $P(A) \neq 0$. The **conditional probability** that the event B occurs, given that the event A occurs, or the **probability of** B **given** A, denoted $P(B|A)$, is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Example 3.11. A card is drawn at random from a standard deck of 52 cards. Find the probability that it is a heart, given that it is red.

Solution. Since we are told that the card is red, there are only 26 cards possible, of which 13 are hearts. Thus,

$$P(\text{heart}|\text{red}) \text{ or } P(H|R) = \frac{13}{26} = \frac{1}{2}$$

■

Example 3.12. A college class consists of 18 students. 10 of the students are men and 8 of the students are women. 6 of the men are freshman and 4 of the women are freshman.

(a) What is the probability that a student selected at random is a freshman?

(b) What is the probability that a student selected at random is a freshman, given that the student is a woman?

Solution. (a) There are 18 students and 10 of them are freshman. So we have

$$P(\text{freshman}) = \frac{10}{18} = \frac{5}{9}$$

(b) If we know that the selected student is a woman and there are 8 women in the class, we know that there are 8 possible outcomes. 4 of the 8 women are freshman. So the conditional probability that the selected student is a freshman, given that the student is a woman is

$$P(\text{freshman}|\text{woman}) = \frac{4}{8} = \frac{1}{2}$$

■

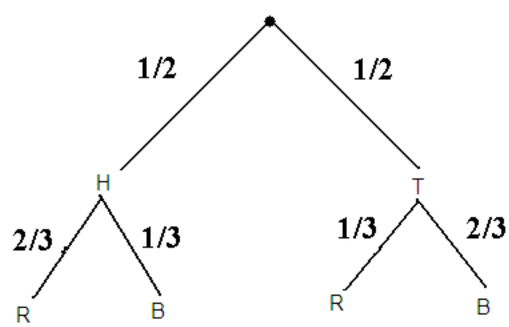
Two events are **independent events** if the occurrence of either event in no way affects the probability of occurrence of the other event. Two events are **dependent events** if the occurrence of one event affects the probability of occurrence of the other event. If A and B are independent events, we can say that

$$P(B|A) = P(B) \quad \text{or} \quad P(A|B) = P(A)$$

Conditional probability can also be found in the case of outcomes that are not equally likely. We will use tree diagrams to make the example more clear.

Example 3.13. We have two jars, each containing three marbles. The first jar contains 2 red marbles and 1 blue marble, and the second jar contains 1 red marble and 2 blue marbles. A fair coin is tossed. If the coin lands heads up, we draw a marble from the first jar. If the coin lands tails up, we draw a marble from the second jar. We need to find the probability that the coin landed tails up, given that a blue marble was drawn.

Solution. We will first construct a tree diagram to describe the experiment. This tree diagram will also shows us the sample space and probabilities.



So we have four outcomes and their probabilities:

$$HR = \frac{1}{2} \times \frac{2}{3} = \frac{2}{6} \qquad HB = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

$$TR = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} \qquad TB = \frac{1}{2} \times \frac{2}{3} = \frac{2}{6}$$

We only want to consider those paths that end with a blue marble being drawn. The sum of the probabilities of the two paths leading to a blue marble is $\frac{1}{6}+\frac{2}{6}=\frac{3}{6}$. So the probability of a blue marble in this experiment is $P(B)=\frac{3}{6}=\frac{1}{2}$. The probability of a tail and a blue marble is $P(T\cap B)=\frac{2}{6}$. So, the probability that the coin lands tails up, given that a blue marble is selected, $P(T|B)$ is as follows:

$$P(T|B) = \frac{P(T \cap B)}{P(B)} = \frac{\frac{2}{6}}{\frac{3}{6}} = \frac{2}{3}$$

■

And Problems

An **and probability problem** requires a successful outcome in *both* of the given events. For example, suppose 2 cards are being randomly selected from a standard deck and we want to find the probability of selecting two kings (one king, followed by a second king). This experiment would only be considered successful if both selected cards are kings.

The *multiplication rule of probability* is used to find the probability that both events A and B occur, represented as $P(A \cap B)$. The **multiplication rule of probability** is

$$P(A \cap B) = P(A) \times P(B|A)$$

Notice if A and B are independent events, then

$$P(A \cap B) = P(A) \times P(B)$$

The following two examples consider an experiment *with* replacement, and then an experiment *without* replacement.

Example 3.14. Two cards are being selected from a standard deck. After drawing the first card, the experimenter returns that card to the deck before drawing their second card. Determine the probability that two aces will be selected.

Solution. A standard deck of 52 cards contains 4 aces, so the probability of selecting an ace on the first draw is $\frac{4}{52}$. That card is then returned to the deck. Thus, the probability of drawing a ace on the second draw is still $\frac{4}{52}$. We will let A be the selection of the first ace and B be the selection of the second ace. We have,

$$\begin{aligned} P(A \cap B) &= P(A) \times P(B|A) \\ &= \frac{4}{52} \times \frac{4}{52} \\ &= \frac{1}{169} \end{aligned}$$

■

Example 3.15. Two cards are being selected from a standard deck. After drawing the first card, the experimenter *does not* return that card to the deck before drawing their second card. Determine the probability that two aces will be selected.

Solution. The probability of selecting an ace on the first draw is still $\frac{4}{52}$. This ace is now discarded and a second ace is drawn from the deck, which now has 51 cards, 3 of which are aces. The probability of selecting an ace on the second draw, then, is $\frac{3}{51}$. We will let A be the selection of the first ace and B be the selection of the second ace. We have,

$$\begin{aligned} P(A \cap B) &= P(A) \times P(B|A) \\ &= \frac{4}{52} \times \frac{3}{51} \\ &= \frac{1}{221} \end{aligned}$$

■

Example 3.14 consisted of independent events, while Example 3.15 consisted of dependent events. As a general rule, *experiments done with replacement will result in independent events, and those done without replacement will result in dependent events*.

Example 3.16. Three hundred people attended the town's charity auction. Only 5 people's names will be drawn to win the door prizes at the end of the auction. These names will be drawn randomly without replacement and each winner will be awarded one door prize. Are the events of selecting the five people independent or dependent events?

Solution. The events are dependent since each time one person is selected, the probability of the next person being selected changes. The probability of the first person being selected is $\frac{1}{300}$. The probability for the next person is then changed to $\frac{1}{299}$. Thus, the events are dependent. ■

Mutually Exclusive Events

Two events A and B are **mutually exclusive** if it is impossible for both events to occur simultaneously. Thus, A and B would have no outcomes in common. Be careful not to confuse mutually exclusive events with independent events. These are not necessarily the same!

Example 3.17. A single card is drawn from a standard 52 card deck. Let B be the event that the card drawn is black, and let D be the event that the card drawn is a diamond.

(a) Are events B and D independent?

(b) Are events B and D mutually exclusive?

Solution. (a) The probability of drawing a diamond is

$$P(D) = \frac{13}{52} = \frac{1}{4}$$

The probability of drawing a diamond, given that the card is black is

$$P(D|B) = 0$$

since if we restrict ourselves to black cards, it is impossible to draw a diamond. This gives us $P(D) \neq P(D|B)$ and hence the two events are dependent.

(b) The probability of drawing a card that is both black and a diamond is

$$P(B \cap D) = 0$$

since it is impossible. Thus the two events are mutually exclusive. ■

Exercises

1. What does the notation $P(A|B)$ mean?

2. What is the formula for $P(A|B)$?

3. If events A and B are mutually exclusive, explain why we are able to simplify the formula
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
to
$$P(A \cup B) = P(A) + P(B)$$

4. A group of friends is selected at random. Let A be the event that one friend plays basketball. Let B be the event that another friend likes to shop.

   (a) Are events A and B mutually exclusive? Explain.

   (b) Are events A and B independent or dependent events? Explain.

For Exercises 5 - 10, consider a situation in which two dice are rolled. Determine the probability that the total on the dice equals

5. 8.
6. 5 if the first die is a 2.
7. 5 if the first die is a 4.
8. a number greater than 8 if the first die is a 6.
9. an odd number if the second die is a 5.
10. a 9 or 11 if the second die is a 3.

For Exercises 11 - 16, let A be the event that a person is a woman and B be the event that a person is a senior. Using words, state what probabilities are expressed by each of the following notations:

11. $P(A|B)$
12. $P(B|A)$
13. $P(\bar{A}|B)$
14. $P(\bar{A}|\bar{B})$
15. $P(A \cap B)$
16. $P(\bar{A} \cap \bar{B})$

17. Suppose a bucket contains 60 marbles, each numbered 1 through 60. Each odd-numbered marble is green and each even-numbered marble is purple. One marble is selected from the bucket and its number and color are recorded.
    (a) List the sample space.
    (b) Consider the events
        A: selecting a green marble
        B: selecting a marble with a number divisible by four
        Find $P(A)$, $P(B)$, $P(A \cup B)$, and $P(A \cap B)$.
    (c) Are A and B mutually exclusive?
    (d) Based on your answer to part 17c, verify the equation for $P(A \cup B)$.

18. Suppose a bucket contains 5 marbles: one yellow, one green, one white, one red, and one blue. Two marbles are selected from the bucket without replacement. Let A be the event the second marble is yellow. Let B be the event the first marble is white. Let C be the event the second marble is green.
    (a) Are events A and B mutually exclusive? Why or why not?
    (b) Are events A and C mutually exclusive? Why or why not?
    (c) Describe the complement of event C.
    (d) Calculate the probability of the event C and the event $\bar{C}$.
    (e) Verify that $P(\bar{C}) = 1 - P(C)$ holds true for the probabilities you calculated in 18d.

19. An experiment requires a fair coin to be tossed and a fair die to be rolled. Consider the following events.

    A: The coin lands heads down.

    B: The die shows a number less than 3.

    (a) Find $P(A)$ and $P(B)$.
    (b) Find and interpret $P(A \cup B)$ and $P(A \cap B)$.
    (c) Are events A and B mutually exclusive?

In Exercises 20 - 23, two cards are drawn at random from a standard 52-card deck. Find the probability of the following

(a) with replacement.
(b) without replacement.

20. They both are aces.
21. The first shows a jack, and the second shows a queen.
22. The first shows a jack or the second shows a queen.
23. Neither shows a face card.

3.4 Expected Value

Expected value or **expectation** is used to determine the expected results of an experiment or business venture over long periods of time. Expectation is used in business to determine profits of a new product. In the insurance industry, expected value is used to determine how much the premiums for a particular insurance policy should be for the company to make an overall profit. Expectation is also used to predict the expected winnings (or losses) in games of chance such as the lottery, roulette, and craps.

Consider the following situation. John tells Alice that he will give her $\$1$ if she tosses a coin and it comes up heads. If the coin comes up tails, she will have to give John $\$1$. We would expect in the long run that half the time John would win $\$1$ and half the time he would lose $\$1$. That is to say, John will break even. Mathematically we find John's expected gain or loss by

$$\begin{aligned}
\begin{array}{c}\text{John's expected}\\ \text{gain or loss}\end{array} &= P\left(\begin{array}{c}\text{John}\\ \text{wins}\end{array}\right)\cdot\left(\begin{array}{c}\text{amount}\\ \text{John wins}\end{array}\right) + P\left(\begin{array}{c}\text{John}\\ \text{loses}\end{array}\right)\cdot\left(\begin{array}{c}\text{amount}\\ \text{John loses}\end{array}\right)\\
&= \frac{1}{2}\times\$1 + \frac{1}{2}\times(-\$1)\\
&= \$0
\end{aligned}$$

Notice that the loss is expressed as a negative number. Since in this case John is expected to break even, we would call this a fair game. If John's expected value is positive, it would indicate a gain; it it were negative, a loss.

The expected value, E, is found by

$$E = P_1 \times A_1 + P_2 \times A_2 + P_3 \times A_3 + \cdots + P_n \times A_n$$

P_1 is the probability the first event will occur and A_1 represents the net amount won or lost if the first event occurs. P_2 is the probability the second event will occur and A_2 represents the net amount won or lost if the second event occurs, and so on.

Example 3.18. For an outdoor concert, concert organizers estimate that $10,000$ people will attend if it is not raining. If it is raining, concert organizers estimate that 8000 people will attend. On the day of the concert, meteorologists predict a 75% chance of rain. Determine the expected number of people who will attend the concert.

Solution.

$$\begin{aligned} E &= P(\text{rain}) \times (\text{attendance if raining}) + P(\text{not rain}) \times (\text{attendance if not rain}) \\ &= 0.75 \times 8000 + 0.25 \times 10,000 \\ &= 8500 \end{aligned}$$

■

Example 3.19. Matt took the AP Calculus exam. On the multiple choice section each question had five possible answers. Matt didn't study, so he randomly selected an answer for every question.

(a) If each correct response earns one point and each incorrect response earns zero points, what percentage of the maximum possible points would Matt expect to receive?

(b) If each correct response earns one point and each incorrect response loses $\frac{1}{4}$ of a point, what percentage of the maximum possible points would Matt expect to receive?

Solution. (a)

$$\begin{aligned} E &= P(\text{correct}) \times (\text{points for correct}) + P(\text{incorrect}) \times (\text{points for incorrect}) \\ &= \frac{1}{5} \times 1 + \frac{4}{5} \times 0 \\ &= \frac{1}{5} \\ &= 0.2 \end{aligned}$$

So Matt would expect to get a score of $0.2 \times 50 = 10$.

(b)

$$
\begin{aligned}
E &= P(\text{correct}) \times (\text{points for correct}) + P(\text{incorrect}) \times (\text{points for incorrect}) \\
&= \frac{1}{5} \times 1 + \frac{4}{5} \times \left(-\frac{1}{4}\right) \\
&= \frac{1}{5} - \frac{1}{5} \\
&= 0
\end{aligned}
$$

So Matt would expect to get a score of $0 \times 50 = 0$.

■

Example 3.20. A game show contestant is offered the option of receiving a computer system worth $\$2300$, or accepting a chance to win either a luxury vacation worth $\$5000$ or a boat worth $\$8000$. If the second option is chosen the contestant's probabilities of winning the vacation and the boat are 0.2 and 0.15, respectively. If the contestant were to turn down the computer system and go for one of the other prizes, what would be the expected winnings? What is the wiser choice?

Solution. If the contestant turns down the computer system, the expected winnings are

$$
\begin{aligned}
E &= P(\text{vacation}) \times (\text{value of vacation}) + P(\text{boat}) \times (\text{value of boat}) \\
&= 0.2 \times \$5000 + 0.15 \times \$8000 \\
&= \$2200
\end{aligned}
$$

From a strictly monetary viewpoint, keeping the computer system is the wiser choice because the value of the computer system is larger than the expected value if the contestant turns down the computer system. ■

Exercises

1. Suppose a multiple choice exam has three possible answers to each question. You get 5 points for each correct answer, but you lose 3 points for each incorrect answer. No points are gained or lost if you leave a question blank.
   (a) What is the expected point value of a random guess?
   (b) What is the expected point value of a guess if the test taker knows with certainty that one of the answers can be eliminated as a possibility and has no preference among the other two?
   (c) Under what conditions should you answer the question?
2. Consider a game that consists of rolling a fair six sided die. You pay $\$5$ to play the game and the $\$5$ is not returned. If you roll a 1, 2, or 3, you win nothing. If you roll a 4 or 5, you win $\$6$; if you roll a 6, you win $\$10$. What is your expected value for this game?

3. Suppose 320 raffle tickets are sold for $\$5$ each. A prize of $\$1000$ will be awarded to one of the ticketholders. Two other ticketholders will be awarded prizes of $\$100$ each. How much should you expect to win or lose on average if you purchase one raffle ticket?

4. An insurance company plans to sell a $\$150,000$ one year life insurance policy to a 38-year-old woman. On the basis of mortality rates for women of her age and background, the insurance company determines the probability of the woman dying in the next year is 0.00104. What should the insurance company charge the woman for the policy if it would like to make an expected profit of $\$45$?

5. Consider a game consisting of rolling a single, fair die. If an even number turns up, you receive that many dollars. If an odd number turns up, you pay that many dollars.

   (a) Find the expected net winnings of this game.

   (b) Is this game fair, unfair against the player, or unfair in favor of the player?

6. A movie theater found in a random survey that 65 customers bought one item at the concession stand, 40 bought two items, 26 bought three items, 14 bought four items, and 18 bought nothing. Find the expected value of items purchased at the concession stand per customer. Round your answer to the nearest tenth.

7. Suppose a contestant on the game show Jeopardy! has hit a Daily Double and decides to wager $\$500$. If the contestant answers the question correctly, she wins $\$500$, and if she answers incorrectly, she loses $\$500$. Based on the category of the question, the contestant thinks she has an 80% chance of answering correctly. What is the expected value of her wager of $\$500$?

8. A man wants to hire a lawyer to handle a $\$100,000$ lawsuit for him. The man must decide between two lawyers. The first lawyer will charge a flat fee of $\$12,000$. The second lawyer will take 30% of the $\$100,000$ if the lawsuit is successful and otherwise gets nothing. With either lawyer, the man thinks he has a 75% chance of winning the lawsuit and collecting $\$100,000$.

   (a) If the man decides to choose the lawyer for which the expected value of the amount he will have to pay is the smallest, which lawyer should he choose?

   (b) Why might the man not choose the lawyer you found in 8a?

9. Suppose you have $\$1000$ to invest and you have two options to choose between. The first option is a guaranteed gain of $\$50$. The second option is somewhat risky, and if it succeeds it will result in a gain of $\$400$. However, there is only a 70% chance it will succeed, and failure will result in a $\$700$ loss. Which investment option should you choose if you want to maximize the expected value of your investment?

10. On a clear day in Boston, the Automobile Association of America (AAA) makes an average of 110 service calls for motorist assistance, on a rainy day it makes an average of 160 service calls, and on a snowy day it makes an average of 210 service calls. If the weather in Boston is clear 200 days of the year, rainy 100 days of the year, and snowy 65 days of the year, determine the expected number of service calls made by the AAA in a given day.

4

Graphs and Networks

One of the classic problems in mathematics is that of the Königsberg bridge problem. In the 1700s, the town was divided into four separate sections which were connected to each other by seven bridges as shown below.

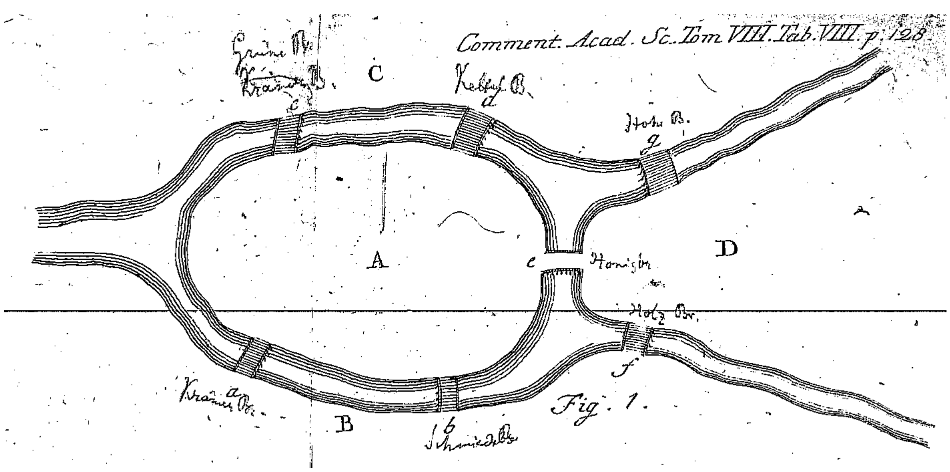


Figure 4.1: Königsberg, circa 1736

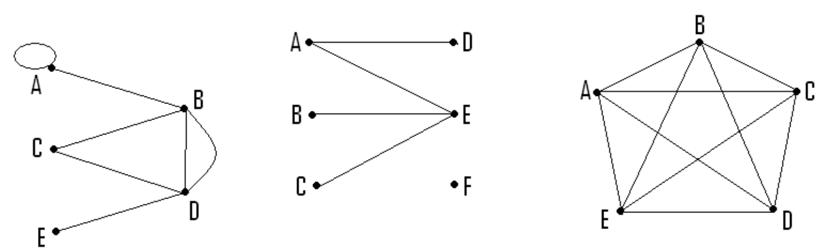
The puzzle was this: Is it possible for a person to walk around town in such a way that each of the seven bridges is crossed once and only once? Leonhard Euler, sometimes referred to as the Shakespeare of mathematics, was visiting Königsberg in 1736 and was presented with this puzzle. Euler solved the puzzle and in doing so laid the foundations for a new type of geometry which he called *geometris situs* or "the geometry of location". These basic ideas of Euler have developed into one of the most important and practical branches of mathematics today – graph theory.

Modern applications of graph theory are included in practically all areas of science and technology from chemistry, biology, and computer science to psychology, sociology and business management.

This chapter discusses the use of graph theory as a tool for solving management science problems. These are problems in which the ultimate goal is to find efficient ways to organize and carry out complex tasks, usually tasks that involve a large number of variables and in which it is not obvious what an optimal solution should be. The first focus will be on how to create efficient routes for the delivery of goods and services such as mail delivery, garbage collection, police patrols, bus routes and the like. These types of problems are known as Euler circuit problems, named after the founder of graph theory, Leonhard Euler. The second focus will be on networks – objects that are somehow connected such as a network of roads that connects cities or a network of fiber optic cable that connects computer servers – and how to find an optimal (or shortest/cheapest) network that connects a given set of objects.

4.1 Graphs

Informally, a **graph** is a picture consisting of dots, called **vertices**, and lines, called **edges**. Edges do not have to be straight lines, but they must connect two vertices. An edge that connects a vertex back to itself is called a **loop**. Figure 4.2 shows a few examples of graphs.

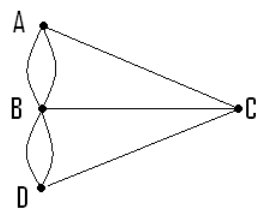


(a) one loop, multiple edges (b) isolated vertex (c)

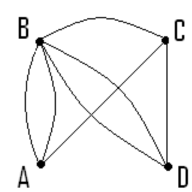
Figure 4.2: Examples of graphs: (a) has a loop at vertex A and multiple edges between vertices B and D, (b) has an isolated vertex at F, (c) the complete graph on five vertices - every possible edge is present with no loops or multiple edges

A precise way to describe a graph is by listing its vertices and edges. The **vertex set** of a graph is a list of its vertices and the **edge set** of a graph is a list of its edges. In Figure 4.2(a), the vertex set is $\mathcal{V} = \{A, B, C, D, E\}$ and the edge set is $\mathcal{E} = \{AA, AB, BC, BD, BD, CD, DE\}$. Edge BD is listed twice because there are two different edges between vertices B and D and AA represents the loop at vertex A. Note that the point in Figure 4.2(c) where edges AC and BE intersect is not a vertex, it just happens that the edges cross there.

A graph depends only on its vertices and edges. Figure 4.3 illustrates two visually different pictures that represent the same graph with vertex set $\mathcal{V} = \{A, B, C, D\}$ and edge set $\mathcal{E} = \{AB, AB, AC, BC, BD, BD, CD\}$.



(a)



(b)

Figure 4.3: Two different representations of the same graph

Terminology

There is a large amount of jargon associated with graph theory. The following are some of the terms that will be used in the remainder of the chapter.

- Two *vertices* are said to be **adjacent** if there is an edge joining them. This has nothing to do with proximity. A pair of vertices that are far apart but have an edge between them are adjacent and two vertices that are close together but have no edge between them are not adjacent. In Figure 4.4, vertices A and B are not adjacent but vertices A and D are adjacent.

- Two *edges* are said to be **adjacent** if they share a common vertex. In Figure 4.4, edges AC and CD are adjacent but edges AC and BD are not adjacent.

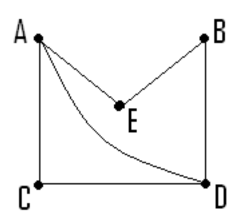
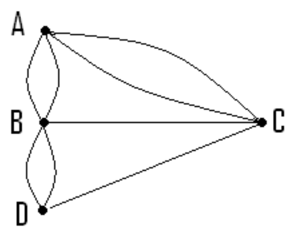


Figure 4.4:



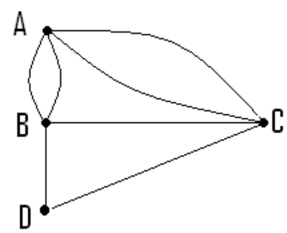


Figure 4.5:

- The **degree of a vertex** is the number of edges at that vertex. When there is a loop at a vertex, the loop contributes twice toward the degree. The notation $\deg(V)$ denotes the degree of vertex V. In Figure 4.4, $\deg(A) = 3, \deg(B) = 2, \deg(C) = 2, \deg(D) = 3, \deg(E) = 2$.

- An **odd** vertex is a vertex of odd degree. An **even** vertex is a vertex of even degree. Figure 4.4 has three even vertices (B, C, and E) and two odd vertices (A and D).

- A **path** can be thought of as a sequence of adjacent edges, where each edge is used only once. The number of edges in the path is called the **length** of the path. An **Euler path** is a path that passes through every edge of the graph. In Figure 4.5(a), A, B, C, D is a path of length 3 from vertex A to vertex D, but it is not an Euler path. In the same figure, $D, B, A, C, B, A, C, D, B$ is an Euler path of length 8.

- A **circuit** is a path that starts and ends at the same vertex. An **Euler circuit** is a circuit that passes through every edge of the graph. In Figure 4.5(b), A, B, C, D, B, A is a circuit of length 5, but it is not an Euler circuit. In the same figure, $D, B, A, C, B, A, C, D,$ is an Euler circuit of length 7.

- A graph is **connected** if, given any two vertices, there is a path joining them. A graph that is not connected is said to be **disconnected**. A disconnected graph is made up of separate connected **components**. Figure 4.6(a) is disconnected with two components and Figure 4.6(b) is connected.

- In a connected graph if there is an edge such that if we were to remove it, the graph would no longer be connected, we say that edge is a **bridge**. In Figure 4.6(b), CG is a bridge.

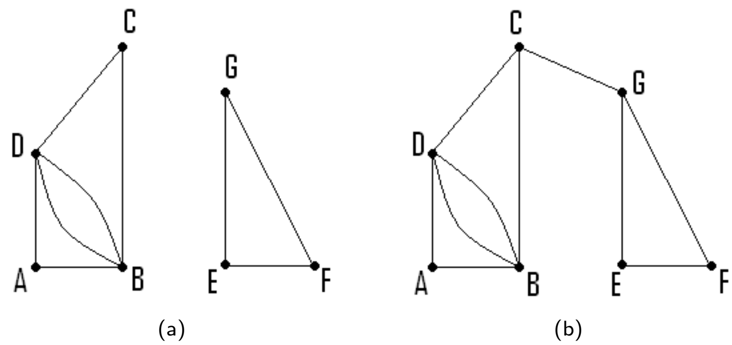
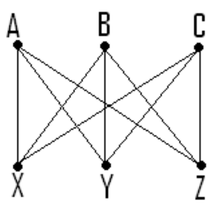


Figure 4.6:

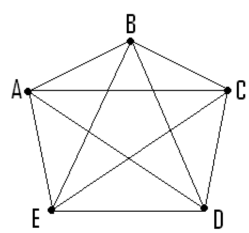
Exercises

In Exercises 1-5, give (a) the vertex set, (b) the edge set, and (c) the degree of each vertex.

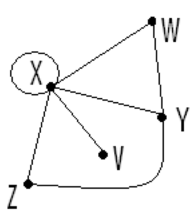
1. .



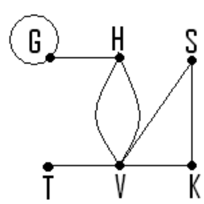
2. .



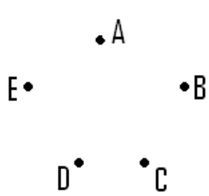
3. .



4. .



5. .



In Exercises 6-9, for the given vertex and edge sets, draw two different representations of the graph.

6. $\mathcal{V} = \{A, B, C, D\}; \mathcal{E} = \{AB, BC, BD, CD\}$
7. $\mathcal{V} = \{K, R, S, T, W\}; \mathcal{E} = \{RS, RT, TT, TS, SW, WW, WS\}$
8. $\mathcal{V} = \{L, M, N, P\}; \mathcal{E} = \{LP, MM, PN, MN, PM\}$
9. $\mathcal{V} = \{A, B, C, D, E\}; \mathcal{E} = \{AC, AE, BD, BE, CA, CD, CE, DE\}$
10. (a) Draw a connected graph with four vertices such that each vertex has degree 2.
    (b) Draw a disconnected graph with four vertices such that each vertex has degree 2.
    (c) Draw a graph with four vertices such that each vertex has degree 1.
11. (a) Draw a connected graph with eight vertices such that each vertex has degree 3.
    (b) Draw a disconnected graph with eight vertices such that each vertex has degree 3.
    (c) Draw a graph with eight vertices such that each vertex has degree 1.
12. Give an example of a graph with four vertices, each of degree three with
    (a) no loops and no multiple edges.
    (b) loops but no multiple edges.
    (c) multiple edges but no loops.
    (d) both multiple edges and loops.
13. Refer to the graph in Figure 4.7 to answer the following.

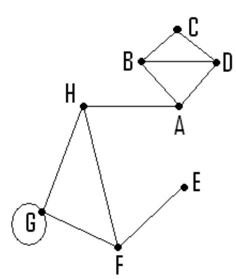


Figure 4.7:

    (a) Find a path from C to F passing through vertex B but not through vertex D.

(b) Find a path from C to F passing through both vertices B and D.

(c) Find a path of length 4 from C to F.

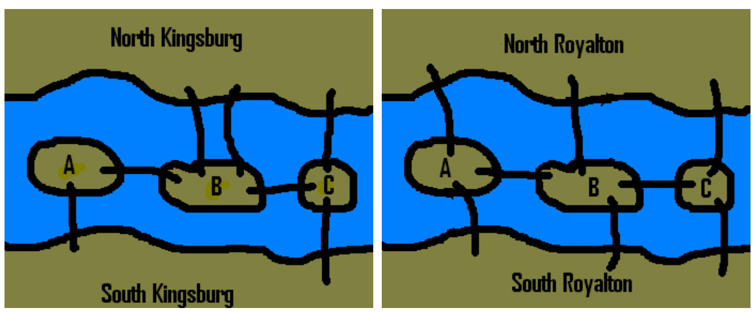
(d) Find a path of length 7 from C to F.

(e) How many paths are there from C to A?

(f) How many paths are there from H to F?

(g) How many paths are there from C to F?

14. Figure 4.8(a) is a map of downtown Kingsburg, showing the Kings River running through the downtown area and the three islands (A, B, and C) connected to each other and both banks by seven bridges. The Chamber of Commerce wants to design a walking tour that crosses all of the bridges. Draw a graph that models the layout of Kingsburg.



(a) Exercise 14 (b) Exercise 15

Figure 4.8:

15. Figure 4.8(b) is a map of downtown Royalton, showing the Royalton River running through the downtown area and the three islands (A, B, and C) connected to each other and both banks by eight bridges. The Downtown Athletic Club wants to design the route for a marathon that passes through the downtown area and through each of the downtown bridges. Draw a graph that models the layout of Royalton.

16. Mr. Belding wishes to make a seating chart for one of his classes at Bayside High. (He wants to minimize the visiting among the students by separating friends as much as possible.) The students in the class are Zack, Screech, Kelly, Lisa, Jessie, Slater, and Tori. Zack is friends with everyone but Slater. Screech is friends with Zack and Slater. Kelly is friends with Zack, Lisa, Jessie, and Slater. Slater is friends with Screech, Kelly, Jessie, and Tori. Draw a graph that Mr. Belding might use to represent the friendship relationships among the students in the class.

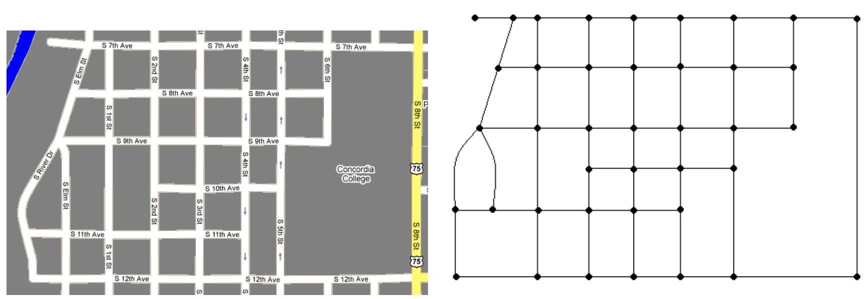
17. The Kangaroo Lodge of Madison County has 10 members ($A, B, C, D, E, F, G, H, I$, and J). The club has five working committees: the Rules Committee (A, C, D, E, I, and J), the Public Relations Committee (B, C, D, H, I, and J), the Guest Speaker Committee (A, D, E, F, and H), the New Year's Eve Party Committee (D, F, G, H, and I), and the Fund Raising Committee (B, D, F, H, and J).

    (a) Suppose we are interested in knowing which pairs of members are on the same committee. Draw a graph that models the situation. (Hint: Let the vertices of the graph represent the members.)

    (b) Suppose we are interested in knowing which committees have members in common. Draw a graph that models this situation. (Hint: Let the vertices of the graph represent the committees.)

4.2 Euler's Theorems

Suppose a police officer has to patrol a neighborhood by foot. The officer must patrol every street in the neighborhood and he would like to start and end at the same spot (where he parks his car). This is actually a graph theory problem! If we treat each intersection as a vertex and each street as an edge, we have a graph that represents the neighborhood. An example of this is shown in Figure 4.9 below.



(a) neighborhood (b) graph of neighborhood

Figure 4.9:

The question we are asking about the patrol officer can be phrased in graph theory terms. Does the graph have an Euler circuit? If it doesn't, how many streets must be re-patrolled? We will begin by answering the question "does this graph have an Euler circuit?"

Euler's Circuit Theorem:

- If a graph is not connected, it does not have an Euler circuit.
- If a graph is connected and every vertex is even, then it has an Euler circuit (at least one, usually more).
- If a graph has any odd vertices, then it does not have an Euler circuit.

The basic idea of Euler's Circuit Theorem is that as we travel along an Euler circuit, each time we cross a vertex we use two edges – one to go into the vertex and one to leave the vertex. This includes the starting point because we leave the vertex in the first step and must come back to it in the last step. Thus if any vertex is odd, at some point we can enter the vertex but are then stuck there because we cannot leave. If a graph is disconnected, there is no way to get from one component to the other. Thus the only possible way to have an Euler circuit is to have a connected graph with all even vertices. We will see in the following section that being a connected graph with every vertex even is enough to guarantee that the graph has an Euler circuit.

The next natural thing to do is ask the question "does this graph have an Euler path?"

Euler's Path Theorem:

- If a graph is not connected, it does not have an Euler path.
- If a graph is connected and every vertex is even, then it has an Euler circuit and hence an Euler path.
- If a graph is connected and has exactly two odd vertices, then it has an Euler path. Any such path will start at one of the odd vertices and end at the other one.
- If a graph has more than two odd vertices, then it cannot have an Euler path.

If we start and end at the same vertex, it becomes an Euler circuit, and we have already answered that question. So if the path starts at one vertex and ends at another we get our result. If the vertex we started at was even, we would have to return to it as many times as we leave it, but since we are ending at a different point, we will leave the vertex one more time than we enter it. Thus the starting vertex must be odd. A similar argument can be made for the ending vertex. All the other vertices must be even so that the number of times we enter and leave are the same. We will see in the following section that being a connected graph with every vertex even or exactly two odd vertices is enough to guarantee that the graph has an Euler path.

The two theorems tell us what happens for disconnected graphs and for connected graphs with zero or two or more odd vertices, but what if a graph has exactly one odd vertex? It turns out that is impossible!

Euler's Sum of Degrees Theorem:

- The sum of the degrees of all the vertices of a graph equals twice the number of edges (and is therefore an even number).
- A graph always has an even number of odd vertices.

This theorem is based on the following. Take any edge in the graph – call it AB. The edge contributes one to the degree of vertex A and one to the degree of vertex B, so in total, it makes a contribution of two to the sum of degrees of a graph. Since each edge contributes two, the sum of the degrees of the vertices of a graph is simply twice the number of edges and hence an even number. Since the sum of the degrees is even, it is impossible to have just one odd vertex, or three odd vertices or five odd vertices – that is, the number of odd vertices must come in pairs.

Putting all of these results together, we know if a graph is disconnected, it has neither an Euler path nor an Euler circuit. Table 4.1 summarizes the results for connected graphs. If we now reconsider the problem of the patrol officer walking

Number of odd vertices	Conclusion
0	Graph has an Euler circuit
2	Graph has an Euler path
$4, 6, 8, \ldots$	Graph has neither
$1, 3, 5, \ldots$	Impossible!

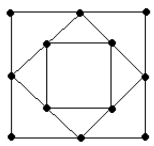
Table 4.1: Summary of Euler's Theorems for Connected Graphs

the neighborhood in Figure 4.9(b), we can see that there is neither an Euler circuit, nor an Euler path for the officer to take because the graph has twenty odd vertices.

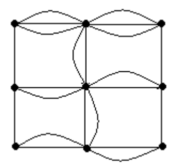
Exercises

Determine whether the graph has an Euler circuit, an Euler path, or neither. Explain your answer. (You do not have to show the actual path or circuit.)

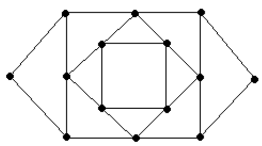
1. .



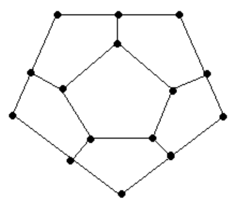
2. .



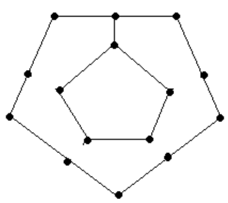
3. .



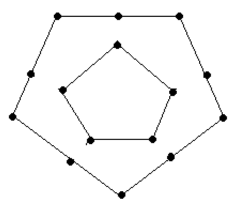
4. .



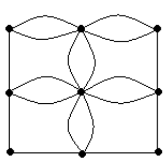
5. .



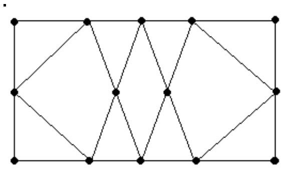
6. .



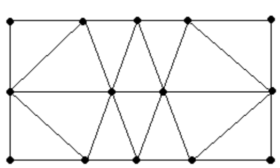
7. .



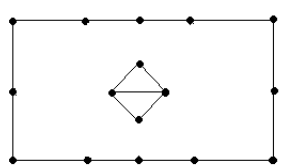
8. .



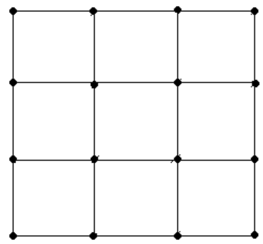
9. .



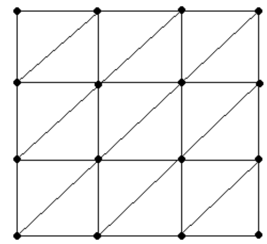
10. .



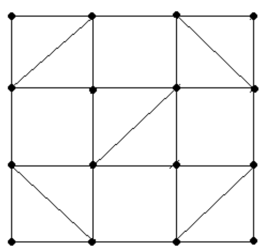
11. .



12. .



13. .



4.3 Fleury's Algorithm

Once we have determined if a graph has an Euler circuit or an Euler path, we might ask the question: if it has an Euler circuit (path), how do we find it? For small graphs, trial-and-error works fine, but for large graphs with thousands of vertices, this would be very time consuming. We need a strategy for finding the path. That is what Fleury's Algorithm does. An *algorithm* is simply a set of instructions, much like a recipe for preparing a meal.

The idea behind Fleury's Algorithm is that bridges should be the last edges you want to cross. Only cross them if there is no other choice. This makes sense because once you have crossed a bridge, you have no way of returning to that component of the graph, so you need to have crossed every edge in that component already.

Fleury's Algorithm for Finding an Euler Circuit (Path):

- **Preliminaries:** Make sure that the graph is connected and has either (i) no odd vertices (circuit) or (ii) exactly two odd vertices (path).
- **Start:** Choose a starting vertex. In case (i) this can be any vertex; in case (ii) it must be one of the odd vertices.
- **Intermediate steps:** At each step, if you have a choice, do not choose a bridge of the yet to be traveled part of the graph. If you have no choice, take it.
- **End:** When you can't travel any more, the circuit (path) is complete. In case (i) you will be back at the starting vertex; in case (ii) you will end at the other odd vertex.

The most complicated part of Fleury's Algorithm is the bookkeeping. One way to keep track of the circuit (or path) is to make two copies of the graph in pencil. On one graph, erase edges as you cross them and on the other graph, trace over them in color and indicate which step you took that edge. Once the first graph is entirely erased, you are done and the second graph has the edges labeled in the order you took them.

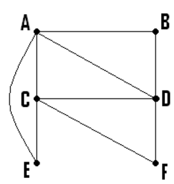
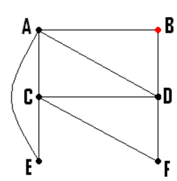


Figure 4.10: Example 4.1

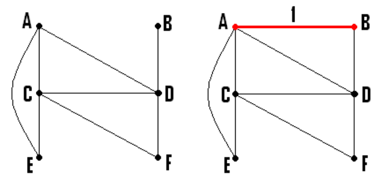
Example 4.1. Apply Fleury's algorithm to the graph in Figure 4.10.

Solution. Note that all the vertices are even, so there will be an Euler circuit. We will show the bookkeeping in every step, as illustrated.

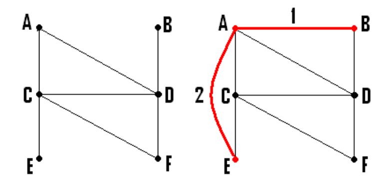
(a) Choose a vertex, say B. Note that any vertex would be okay.



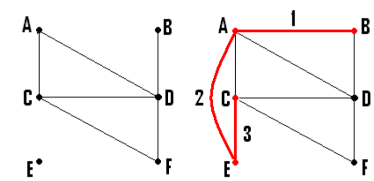
(b) Then travel from B to A. We also could have gone from B to D.



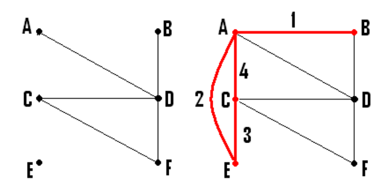
(c) Travel from A to E. We could have also gone to C or D.



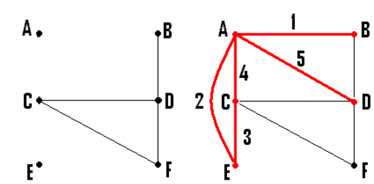
(d) Travel from E to C.



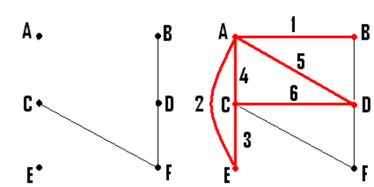
(e) Travel from C to A. We could have also gone to D or F.



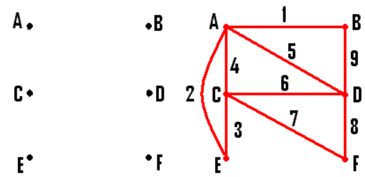
(f) Travel from A to D.



(g) Travel from D to C. We could have also gone to F but not to B since it is a bridge.



(h) Travel from C to F to D to B.

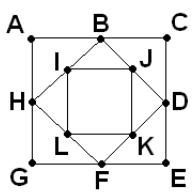


Thus an Euler circuit in the graph is $BAECADCFDB$. ∎

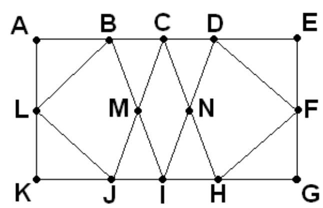
Exercises

In Exercises 1-4, find an Euler circuit.

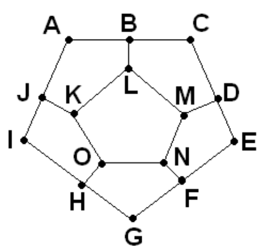
1. .



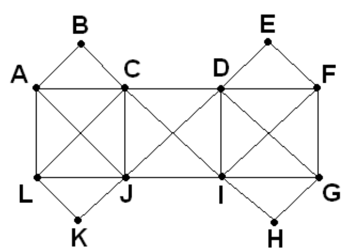
2. .



3. .

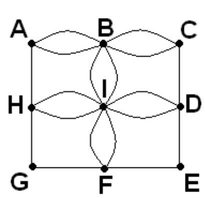


4. .

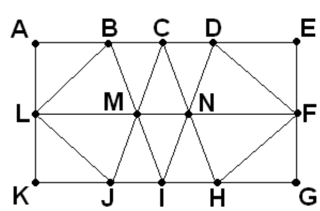


In Exercises 5-8, find an Euler path.

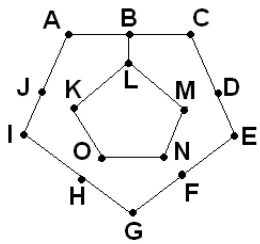
5. .



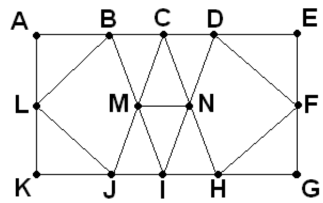
6. .



7. .



8. .



9. In his original paper on the Königsberg bridge problem, Euler posed the following question: Let us take an example of two islands with four rivers forming the surrounding water. There are fifteen bridges across the water around the islands and the adjoining rivers, as in Figure 4.11. The question is whether a journey can be arranged that will pass over all the bridges but not over any of them more than once. Give an answer to Euler's question. If the journey is possible, describe it. If it isn't, explain why not.

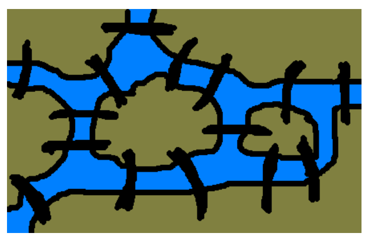


Figure 4.11: Exercise 9

4.4 Eulerizing Graphs

If we have determined that a connected graph does not have an Euler circuit or an Euler path, a common question is: "is there an optimal exhaustive route in the graph?" Stated another way, what is the fewest number of edges needed to traverse every edge in the graph at least once? This question is practical because in many applications each edge represents some unit of cost – a distance traveled, fuel used or time spent – and we would like to minimize the cost we pay.

The key idea in solving this problem is to turn odd vertices into even vertices by adding duplicate copies of existing edges in strategic places. This process is called **eulerizing** the graph.

Example 4.2. The graph in Figure 4.12 represents a 3 block by 3 block street grid consisting of 24 blocks. Find an optimal exhaustive route to cover the grid that begins and ends at the same vertex.

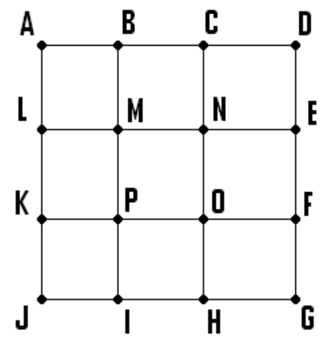


Figure 4.12: Example 4.2

Solution. Our first step is to identify the odd vertices. The graph has eight odd vertices: B, C, E, F, H, I, K, and L. When we add a duplicate copy of edges BC, EF, HI, and KL, we get the graph in Figure 4.13. This is an eulerized version

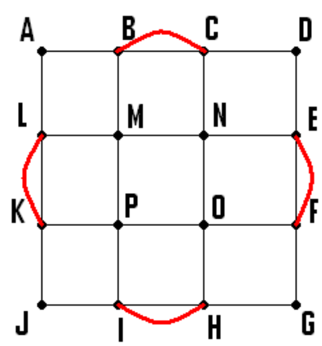


Figure 4.13: Eulerized version of Figure 4.2

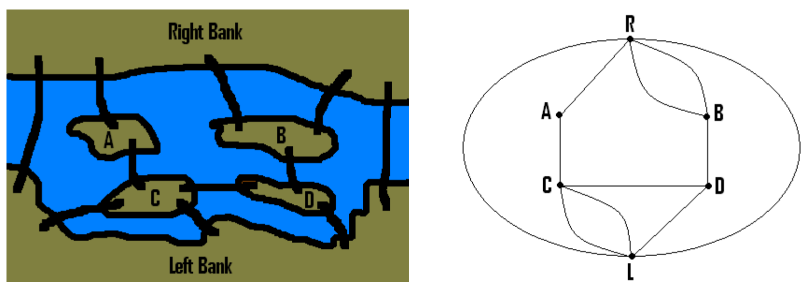
of the original graph – its vertices are all even, so it must have an Euler circuit. It should be clear that we couldn't have done this with fewer than four edges. Using Fleury's Algorithm we can then find an Euler circuit. One such possibility is

$$ABCNOHIPMBCDEFOPKLMNEFGHIJKLA$$

where we recross the edges BC, EF, HI, and KL. The total length of this route is 28 blocks (the 24 blocks in the grid plus the four edges that were recrossed). This route is optimal – the minimum number of edges that need to be recrossed is 4, and we must cross the original 24 blocks, so the minimum is 28 blocks. ■

In some situations we need to find an exhaustive route, but there is no requirement that we begin and end in the same spot. In these cases we want to leave two odd vertices – one as the starting point and one as the ending point – and change the other odd vertices into even ones by adding duplicate edges. This process is called a **semi-eulerization** of the graph. If we add duplicate edges in a strategic manner, we can obtain an optimal exhaustive open route.

Example 4.3. Madison County is a quaint old place, famous for its quaint old bridges. A river runs through the county and there are four islands and eleven bridges joining the islands to the banks of the river and to one another as in Figure 4.14(a). A photographer is hired to take pictures of each of the bridges for a magazine. There is a $\$10$ toll every time a visitor drives across a bridge. The photographer wants to minimize the total cost of her trip and to recross the bridges only if absolutely necessary. What is minimum amount of money she will spend on tolls if she must begin on the right bank and end on the left bank?



(a) (b)

Figure 4.14: Example 4.3

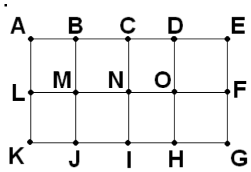
Solution. The graph in Figure 4.14(b) is a representation of the situation and we need to find an optimal exhaustive open route.

Note that the graph has four odd vertices (R, L, B, and D), so it does not already have an Euler circuit or path. Since we want to begin at R and end at L, we do not need to add any edges for those vertices. Thus we only need to add edges to make B and D even vertices. If we add a duplicate edge BD, that will accomplish our goal. Thus our path would cross all of the edges in the graph once except for BD which it would cross twice. This amounts to crossing bridges 12 times, which would cost the photographer $\$120$. ■

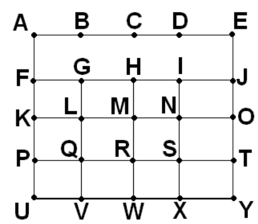
Exercises

In Exercises 1-4, find an optimal eulerization.

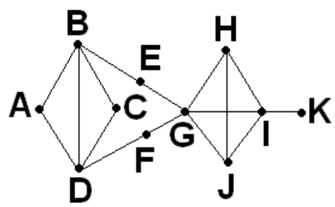
1. .



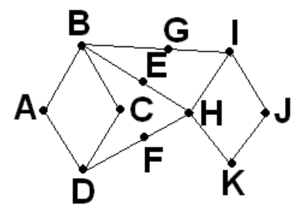
2. .



3. .

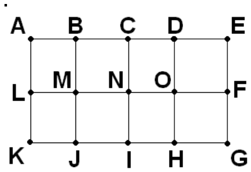


4. .

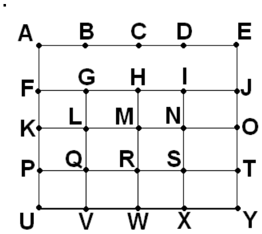


In Exercises 5-8, find an optimal semi-eulerization.

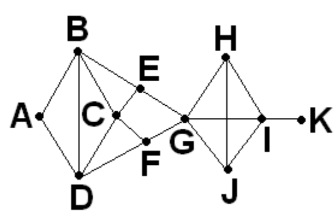
5. .



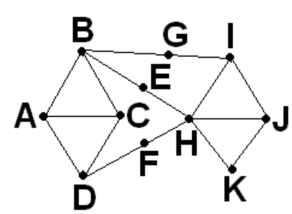
6. .



7. .



8. .



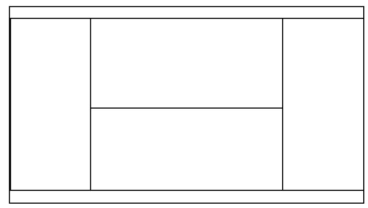


Figure 4.15: Exercise 9

9. How many times would you have to lift your pencil to trace the diagram of the tennis court in Figure 4.15, assuming that you do not trace over any line segment twice? Explain. (This is the problem facing a groundskeeper trying to mark the chalk lines of a tennis court.)

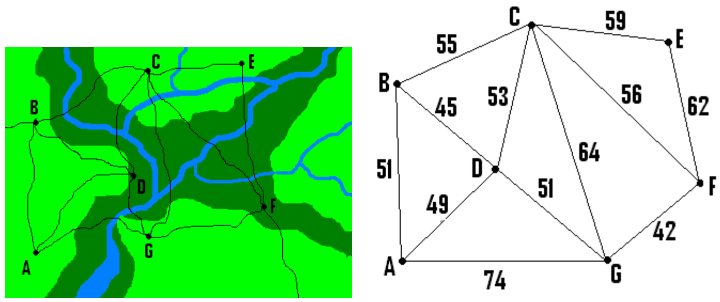
4.5 Networks

A **network** is a graph that is connected. Typically the vertices are "objects" such as cities, computer servers, human beings, and so on, while the edges indicate connections among the objects like roads, internet connections, social connections, and so on.

The most well known network is the World Wide Web. It is a classic example of an *evolutionary* network – it follows no predetermined master plan and evolves on its own without structure. Most social animals belong to many different overlapping evolutionary networks. On the other end of the spectrum are networks that are centrally planned and designed to meet certain goals like roads, fiber-optic cable lines, power lines, and so on. These networks are usually expensive to build and one of the primary considerations when building these networks is minimizing their cost. This section focuses on the problem of finding optimal networks. In our context, an optimal network is the cheapest or shortest network.

Trees

An example will illustrate the concepts we are interested in. The Acme Telephone Company is contracted to provided telephone, cable and internet service to seven small mountain towns shown in Figure 4.16(a). These towns are located in a region where cell phone towers and wireless communication are out of the question. The terrain is so rugged, the company would like to bury the underground cables along existing roads. The network of roads is shown in Figure 4.16(a) and the weighted graph in Figure 4.16(b) describes the situation. The vertices represent the towns, the edges represent the roads between the towns and the weight of each edge represents the cost (in millions of dollars) of putting a fiber-optic cable connection



(a) (b)

Figure 4.16: Acme Phone Company

along the road. The basic question is: Find a network that (a) utilizes the existing network of roads; (b) connects all of the towns; and (c) has the least cost.

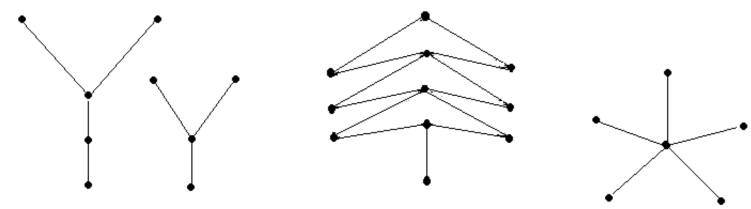
In the terminology we have developed, the network we are looking for

- is a *subgraph* of the original graph – its edges must come from the original graph.
- must *span* the original graph – it must include all the vertices of the original graph.
- must be *minimal* – the total weight of the network should be as small as possible.

The third bullet point has an important implication. A minimal network cannot have any circuits. If A and B are two vertices in a network and we have a path from A to B through some other vertices, it would be redundant to include an edge directly from A to B. (In real-life applications, sometimes we will want redundant connections, but for our purposes we exclude them.)

The basis of a minimum network is a type of graph called a *tree*. A **tree** is a network with no circuits. Figure 4.17 compares graphs that are trees with graphs that are not. A **spanning tree** of a network is a subgraph that connects all the vertices of the network and has no circuits. A **minimum spanning tree (MST)** is a spanning tree of a network with least total weight.

Trees occupy an important place between disconnected graphs and "overconnected" graphs. We can think of trees as being barely connected. This gives us several properties of trees.



(a) Not a tree – not connected (b) Not a tree – contains circuits (c) Tree

Figure 4.17:

Properties of Trees:

Property 1

- In a tree, there is one and only one path joining any two vertices.
- If there is one and only one path joining any two vertices of a graph, then the graph must be a tree.

Property 2

- In a tree, every edge is a bridge.
- If every edge of a network is a bridge, then the network must be a tree.

Property 3

- A tree with N vertices has $N - 1$ edges.
- If a network has N vertices and $N - 1$ edges, then it must be a tree.

Spanning Trees

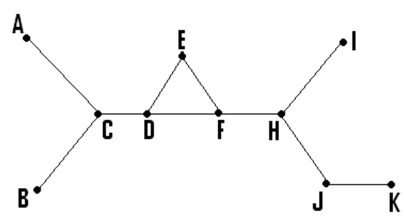
We may generalize Property 3 to obtain another property of trees:

Property 4

- If a network has N vertices and M edges, then $M \geq N-1$. We will refer to the difference $R = M - (N-1)$ as the **redundancy** of the network.
- If $M = N - 1$, the network is a tree; if $M > N - 1$, the network has circuits and is not a tree. In other words, a tree is a network with zero redundancy and a network with positive redundancy is not a tree.

If a network has positive redundancy, there are many trees within the network that connect all of its vertices. These are the spanning trees of the network.

Example 4.4. Find the spanning trees of the following network.



Solution. There are $N = 10$ vertices and $M = 10$ edges, so the redundancy of the network is

$$R = M - (N - 1) = 10 - (10 - 1) = 1$$

To find a spanning tree we have to "discard" one edge. Seven of the edges are bridges, so they must be part of any spanning tree. The other three edges (DE, EF, and DF) form a circuit of length 3, so if we exclude any one of these three edges, we will have a spanning tree. Thus the network has three spanning trees, see Figure 4.18. ■

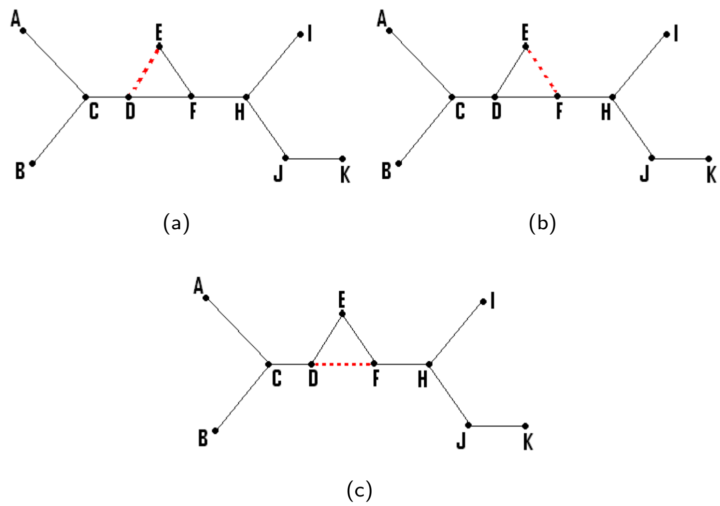


Figure 4.18: Example 4.4

Example 4.5. Find the number of different spanning trees of the network in Figure 4.19.

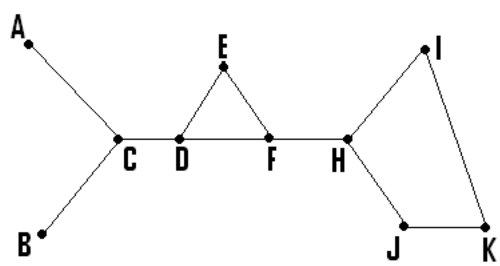


Figure 4.19: Example 4.5

Solution. There are $N = 10$ vertices and $M = 11$ edges, so the redundancy of the network is

$$R = M - (N - 1) = 11 - (10 - 1) = 2$$

To find a spanning tree we have to “discard” two edges. Four of the edges are bridges, so they must be part of any spanning tree. The other seven edges form two circuits: DE, EF, and DF form a circuit of length 3 and HI, IK, KJ, and HJ form a circuit of length 4. So if we exclude one edge from each of these two circuits, we will have a spanning tree. There are 3 possible edges to discard from the first circuit and four possible edges to discard from the second circuit. Thus the network has $3 \times 4 = 12$ spanning trees. ■

These examples lead us to a strategy for finding minimum spanning trees. In a weighted network, we look for spanning trees as in the previous examples. Since our goal is to have a spanning tree with the smallest possible weight, we will "discard" edges as in the examples with the added idea that we would like to discard the most expensive edges. However, this strategy becomes extremely difficult to implement if the network is large or very complicated.

Kruskal's Algorithm

Fortunately, there are several algorithms for finding minimum spanning trees. One of the nicest of these is Kruskal's Algorithm. The idea is to build a minimum spanning tree one edge at a time, choosing at each step the cheapest available edge. The only restriction on our choices is that we should never choose an edge that creates a circuit. The remarkable thing about Kruskal's Algorithm is that it always yields an optimal solution!

Kruskal's Algorithm:

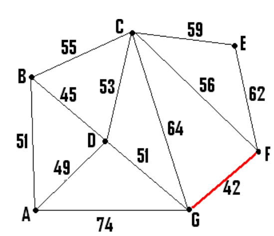
- **Step 1:** Pick the cheapest edge available. In the case of a tie, choose one at random. Mark it.
- **Step 2:** Pick the next cheapest link available (one that does not form a circuit) and mark it. In the case of a tie, choose one at random.
- **Step** $3, 4, \ldots, N-1$**:** Continue picking and marking the cheapest available unmarked link.

This algorithm is as good as algorithms get – it always produces an optimal solution and it is very efficient. No matter how complicated the network, Kruskal's Algorithm will always find a minimum spanning tree in an efficient manner.

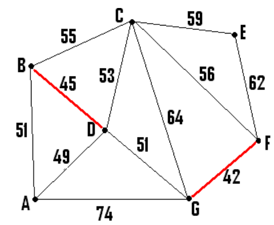
Example 4.6. Apply Kruskal's Algorithm to the Acme Phone Company problem presented at the beginning of the section and modeled by Figure 4.16(b). What is the minimum cost for laying the cable?

Solution. .

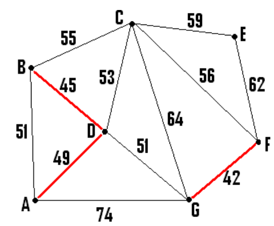
Step 1: Choose the cheapest available edge – in this case GF which costs $\$42$ million. Mark the edge, say in red on the graph.



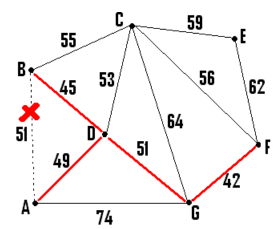
Step 2: The next cheapest edge available is BD at \$45 million. Mark the edge.



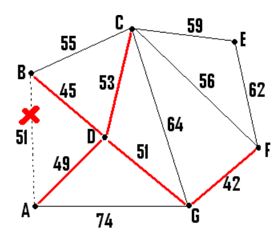
Step 3: The next cheapest edge available is AD at \$49 million. Mark the edge.



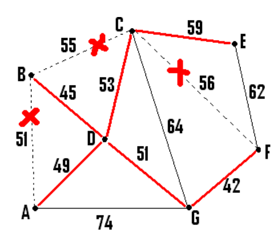
Step 4: There is a tie for the next cheapest edge – both AB and DG are \$51 million. But AB would form a circuit in the graph and so we cannot choose it. Thus we would choose DG and mark it.



Step 5: The next cheapest edge available is CD at \$53 million. Mark the edge.



Step 6: The next cheapest edge is BC at \$55 million, but it forms a circuit, so we cannot choose it. The next possible choice is CF at \$56 million, but it also forms a circuit. The next possible choice is CE at \$59 million and it is okay. Mark the edge.

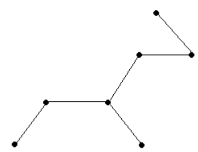


Since a spanning tree must have $M = N - 1$, we know we are finished and have a minimum spanning tree! ■

Exercises

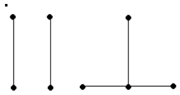
In Exercises 1-4, determine whether the graph is a tree or not. If not, explain why not.

1. .

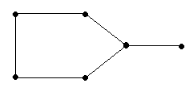


2. .

3. .



4. .



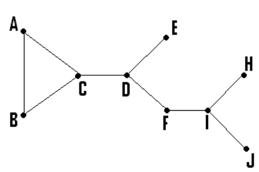
In Exercises 5 and 6, assume that G is a graph with no loops or multiple edges, and choose the option that best applies: (i) G is definitely a tree (explain why); (ii) G is definitely not a tree (explain why); (iii) G may or may not be a tree (give two examples of graphs that fit the description – one a tree and the other one not).

5. (a) G has 8 vertices and 10 edges.
   (b) G has 8 vertices and 7 edges.
   (c) G has 8 vertices and 7 bridges.

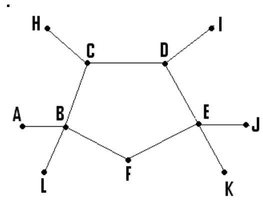
6. (a) G has 8 vertices and there is exactly one path from any vertex to any other vertex.
   (b) G has 8 vertices and no bridges.
   (c) G has 8 vertices, is connected, and every edge in G is a bridge.

In Exercises 7 and 8, find all the spanning trees of the network.

7. .

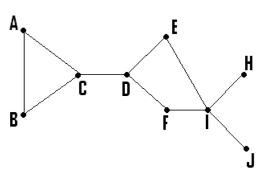


8. .

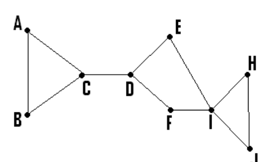


In Exercises 9-12, find the number of different spanning trees.

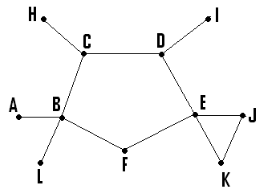
9. .



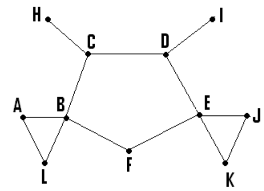
10. .



11\. .

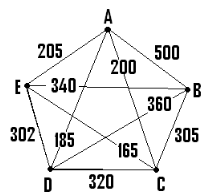


12\. .

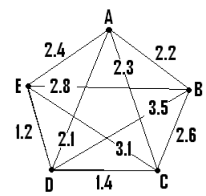


In Exercises 13 and 14, (a) find the MST for the network using Kruskal's Algorithm and (b) give the weight of the MST found in (a).

13\. .



14\. .



15\. The following mileage chart shows the distances between Atlanta (A), Columbus (C), Kansas City (KC), Minneapolis (M), Pierre (P), and Tulsa (T). Working directly from the mileage chart, implement Kruskal's Algorithm and find the minimum spanning tree connecting the cities.

	A	C	KC	M	P	T
A	$*$	533	798	1068	1361	772
C	533	$*$	656	713	1071	802
KC	798	656	$*$	447	592	248
M	1068	713	447	$*$	394	695
P	1361	1071	592	394	$*$	760
T	772	802	248	695	760	$*$

5

Consumer Mathematics

Do you know the difference between simple and compound interest? What about your credit cards? How much interest do you pay on them? Do you want to own a home someday? How much of a down payment should you make and what type of loan should you look for? And what are closing costs? As an informed consumer, you should be aware of these topics. This chapter introduces a few of the things you need to know. For a brief review of percentages see Chapter 1.

5.1 Interest

Lending and borrowing money is a part of daily life. For instance, an automobile costs $\$20,000$. The bank will loan you the money and in return you will pay the bank $\$22,000$. The additional fee added to the original loan, in this case $\$2,000$, is known as **interest**.

Thus when lending or depositing money, you receive interest on its use. On the other hand, when borrowing, you pay the institute or person interest until your debt is repayed.

When computing interest you must look at three figures: principal, rate, and time. The **principal** is the original amount you deposit or borrow. The **rate**, or the **rate of interest**, is the percentage of principal that is paid, per year, for the use of the money. Finally, **time** is the duration of the deposit or loan, in *years*.
In the example above, the bank used simple interest to find the fee or interest. However, there is more than one type of interest. The two types of interest are: *simple interest* and *compound interest*.

Simple Interest

Simple interest is the interest found by computing the product of the principal, the rate of interest, and the duration, or time, that the money is borrowed/loaned.

$$I = Prt$$

where I is the interest, P is the principal, r is the rate of interest, and t is the time (in years) until repayment.

Example 5.1. An automobile costs $\$20,000$. The Northern Bank of America will loan you the money at a 1% interest rate.

(a) Find the interest after one year.

(b) Find the interest after ten years.

Solution. For both problems we will use the simple interest formula with $P = \$20,000,\ r = 1\% = 0.01,$ and $t =$ given.

(a) $I = Prt = \$20,000 \times 0.01 \times 1 = \$200$

(b) $I = Prt = \$20,000 \times 0.01 \times 10 = \$2,000$

By changing the number of years from one to ten, we see that the amount of interest increased to $\$2,000$. ■

We have now shown that in ten years the bank will receive $\$2,000$ in interest from a loan of $\$20,000$. Thus, you will have to pay $\$22,000$ for the car. This total is called the **future value**, sometimes called **maturity value**. The future value (A, which stands for "amount") is found by computing the principal, also called **present value** (P), plus the interest (I). The future value formula is as follows:

$$\text{future value} = \text{principal} + \text{interest} = P + Prt = P(1 + rt)$$

Example 5.2. A deposit of $\$5,000$ has been made into a savings account with a rate of 2.5%. Find the loan's future value after 3 months.

Solution. The amount borrowed, or the principal, P, is $\$5,000$. The rate, r, is 2.5%, or 0.025. The time, which needs to be represented in years, is $\frac{1}{4}$ or 0.25 years.

The loan's total amount after three months is:

$$A = P(1 + rt) = \$5,000[1 + (0.025)(0.25)] = \$5,031.25$$

Therefore, the loan's future value after three months is $\$5,031.25$. ■

Simple interest loans are usually thought of as short-term loans, like automobile and consumer loans. Now, you have calculated interest from given principals, but did you know you can also find the principal, rate, or time of a loan by using the formula $A = P(1 + rt)$ and the given information?

Example 5.3. Suppose that Amanda is granted a 10 month deferral on a $\$410$ payment rather than a loan. That is, she will have to pay $\$410$ at the end of the 10 month period. Amanda has extra money she wants to put into her savings account. If she can earn 4% simple interest on her savings account, how much money must she deposit now so that its value will equal $\$410$ in 10 months?

Solution. The future amount, A, is $\$410$. The rate, r, is 4% or 0.04. The time is 10 months, which equals $\frac{10}{12}$. We need to use $A = P(1 + rt)$ to find the principal.

$$
\begin{aligned}
P\left[1 + (0.04)\left(\frac{10}{12}\right)\right] &= 410 \\
P &= \frac{410}{1.0\overline{3}} \\
P &\approx \$396.77
\end{aligned}
$$

Amanda will need to deposit $\$396.77$ into her savings account that has a 4% interest rate for ten months in order to pay back her loan of $\$410$. ■

Example 5.4. Eric needs to borrow $\$5,000$ for a new TV and audio surround system. The bank agrees to lend Eric the money for 7 years if he agrees to pay the bank a total of $\$5,500$ at the end of the 7 years. Find Eric's interest rate.

Solution. The principal amount, P, is $\$5,000$. The future amount, A, is $\$5,500$. The time is 7 years. Again we will use the future value formula, $A = P(1 + rt)$.

$$
\begin{aligned}
(\$5,000)[1 + r(7)] &= \$5,500 \\
\$5,000 + \$35,000r &= \$5,500 \\
\$35,000r &= \$500 \\
r &\approx 0.014 \\
r &\approx 1.4\%
\end{aligned}
$$

From the calculations, we found that Eric is receiving a 1.4% interest rate on his loan. ■

Compound Interest

In 1626 the Wappinger Indians sold Peter Minuit Manhattan Island for $\$24$. If the natives would have put that $\$24$ in the bank with simple interest at a rate of 5%, in 2006 it would be worth $\$24 \times 380 \times 0.05 = \$456$. However if the natives would have put the money into a savings account with *compound interest*, the future value in 2006 would be over $\$4$ billion dollars.

Compound interest is the interest earned on the original amount plus the accumulated interest. In other words, after a certain period, the interest earned up to that point is credited to the account and is added to the current sum. This new amount then earns interest during the next period, and repeats every period.

So, let's say the Wappinger Indians put their $\$24$ into a compound interest savings account with a 5% interest rate.

Year	Current Balance	Formula for Year's End
1	$\$24$	$A = \$24(1+0.05) = \$25.20$
2	$\$25.20$	$A = \$25.20(1+0.05) = \$26.46$ $= \$24(1+0.05)(1+0.05) = \$24(1+0.05)^2$
3	$\$26.46$	$A = \$26.46(1+0.05) = \$27.78$ $= \$24(1+0.05)^2(1+0.05) = \$24(1+0.05)^3$

We can now deduce that the **compound formula** used for yearly compounding is a modified version of the future value formula. Like in the last section, A is the amount in the account, P is the present value of the account, r is the interest rate, and t is the time in years.

$$A = P(1+r)^t$$

The time between the interest periods is called the **compounding period**. The example above had its interest **compounded annually**, or compounded once a year.

Example 5.5. Find (i) the future value and (ii) the amount of interest earned for the following deposits.

(a) $\$10,450$ at 5% compounded annually for 4 years.

(b) $\$4,690$ at 3.5% compounded annually for 28 months.

Solution. We will use the compounded annually formula $A = P(1+r)^t$ as follows:

(a) (i) We know $P = \$10,450$, $r = 0.05$, and $t = 4$. When put into the formula,

$$A = \$10,450(1+0.05)^4 = \$12,702.04$$

we see that the future value is $\$12,702.04$ after four years of annual interest.

(ii) To find the interest earned we subtract the present value from the future value,

$$I = \$12,702.04 - \$10,450 = \$2,252.04$$

and find that the interest is $\$2,252.04$.

(b) (i) First we see that $P = \$4,690$, $r = 0.035$, and $t = \frac{28}{12}$. When put into the formula,

$$A = \$4,690(1+0.035)^{\frac{28}{12}} = \$5,081.99$$

we see that the future value is $\$5,081.99$ after 28 months of annual interest.

(ii) The interest is again found by subtracting the new amount from the previous amount.

$$A = \$5,081.99 - \$4,690 = \$391.99$$

We found that the interest gained on this account was $\$391.99$ $\blacksquare$.

An account can also be compounded more than once a year. For instance, twice a year is called, **compounded semiannually**. Four times a year, **compounded quarterly**. In general, the number of times an account is compounded is referred to as n **compounding periods per year**. The general formula for multiple compounds a year is:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where n is the number of compounds per year.

Example 5.6. The Blinkers' son will put $\$10,000$ in his savings account for 4 years to help pay for his college education. What lump sum will be produced after the four years if the Blinkers' son deposited today at a 4% rate compounded quarterly?

Solution. We will use the multiple compound formula, $A = P\left(1 + \frac{r}{n}\right)^{nt}$ to compute our answer. We know that $P = \$10,000$, $r = 0.04$, $n = 4$, and $t = 4$. Thus,

$$A = \$10,000\left(1 + \frac{0.04}{4}\right)^{4\times4} = \$11,725.79$$

So, at the end of four years, the Blinkers' son will have $\$11,725.79$ towards his college education. $\blacksquare$

Example 5.7. Colleen has a bank account with an 8% interest rate that compounds her savings daily. If she put $\$2,975$ into the bank for 3 years, what would be her future value?

Solution. Again we will use $A = P\left(1 + \frac{r}{n}\right)^{nt}$ to compute Colleen's total after three years.

$$A = \$2,975\left(1 + \frac{0.08}{365}\right)^{365\times3} = \$2,975(1.000219)^{1,095} = \$3,781.87$$

So we see that after three years of daily compounding, Colleen's future value increased to $\$3,781.87$ $\blacksquare$.

Some banks also use **continuous compounding**, where the compounding periods occur every trillionth of a second, quadrillionth of a second and so on. So, as n increases to infinity, the expression $\left(1 + \dfrac{1}{n}\right)^{n}$ becomes close to the irrational number e, where $e \approx 2.71828$. Because the expression approaches e, the general formula for continuous compounding becomes,

$$A = Pe^{rt}$$

where A is the future amount, P is the present amount, r is the interest rate, and t is time.

Example 5.8. You decide to invest $\$10,000$ for 10 years and you have a choice between two accounts. The first pays 9% per year, compounded monthly. The second pays 8.65% per year and is compounded continuously. Which investment is a better choice for you?

Solution. The better investment will give you a larger balance after 10 years. Let's begin with the account with monthly compounding.

1. By using the multiple compounding formula, $A = P\left(1+\frac{r}{n}\right)^{nt}$, we see that $P = \$10,000,\ r = 0.09,\ n = 12$, and $t = 10$. Next we plug these numbers into the formula

$$A = \$10,000\left(1+\frac{0.09}{12}\right)^{12\times\ 10} = \$10,000(1.01)^{120} = \$24,513.57$$

So we see that at the end of 10 years, $A = \$24,513.57$.

2. Now let's look at continuous compounding. We will use the formula $A = Pe^{rt}$, where $P = \$10,000,\ r = 0.0865$, and $t = 10$. Next we plug these numbers into the formula

$$A = \$10,000e^{0.0865\times 10} = \$10,000e^{0.865} = \$23,750.06$$

Thus with continuous compounding, $A = \$23,750.06$.

Comparing the two accounts, notice that the monthly compounding results in a larger future balance. Therefore you should choose the account with monthly compounding. ∎

In the multiple compounding formula, the unknown quantity could be time, which appears in the exponent. To solve for an unknown in the exponent, we use logarithms. For a brief review of logarithms, see Chapter 1.

Example 5.9. You are anxious to know when your money will double in your savings account that earns 3% interest and compounds daily. Find when your money doubles.

Solution. Use the multiple compounding formula where your future amount will be twice the value of your present value,

$$\begin{aligned}
2P &= P\left(1+\frac{r}{n}\right)^{nt} \\
2P &= P\left(1+\frac{0.03}{365}\right)^{365t} \\
2 &= \left(1+\frac{0.03}{365}\right)^{365t} \\
2 &= (1.0000822)^{365t} \\
\ln 2 &= 365t\ln(1.0000822) \\
\frac{\ln 2}{\ln 1.0000822} &= 365t \\
8432.79 &\approx 365t \\
\frac{8432.79}{365} &\approx t \\
23.10 &\approx t
\end{aligned}$$

Thus, your money doubles about 23.1 years after you first put the money in the bank. ■

Effective Annual Yield

Besides the actual annual interest rate, called the **nominal rate**, many savings institutions will advertise an **effective annual yield**. The effective annual yield of a compound interest deposit is the simple interest rate that has the same future value as the compound rate would have after one year.

Example 5.10. You deposit $\$3,000$ in an account that pays 7% interest compounded monthly.

(a) Find the future value after one year.

(b) Use the future value formula to find the effective annual yield.

Solution. (a) Use the compound interest formula to find the future value of $\$3,000$.

$$A = P\left(1+\frac{r}{n}\right)^{nt} = \$3,000\left(1+\frac{0.07}{12}\right)^{12\times 1} \approx \$3,216.87$$

(b) To find the effective annual yield, use the simple interest rate. So, we look at the future value formula used to find simple interest.

$$
\begin{aligned}
A &= P(1+rt) \\
\$3,216.87 &= \$3,000(1+1r) \\
\$3,216.87 &= \$3,000+\$3,000r \\
\$216.87 &= \$3,000r \\
r &= 0.07229 \\
&= 7.2\%
\end{aligned}
$$

Therefore, this bank will advertise a compound interest rate of 7% and a simple interest rate of 7.2% ■.

The effective annual yield is often used to compare accounts and banks. The better the effective annual yield the better the investment. When borrowing money, the effective rate is called **annual percentage rate** or **APR**. The general formula for finding the effective annual yield on the nominal rate is

$$Y = \left(1+\frac{r}{n}\right)^n - 1$$

where r is the nominal rate and n is the number of compounds per year.

Example 5.11. A savings account has a nominal rate of 6%. The interest is compounded daily. Find the account's effective annual yield.

Solution. The rate, r, is 0.06. Because the interest is compounded daily, $n = 365$. The account's effective annual yield is

$$Y = \left(1+\frac{r}{n}\right)^n - 1 = \left(1+\frac{0.06}{365}\right)^{365} - 1 \approx 0.0618 = 6.18\%$$

The effective annual yield is 6.18%. Thus, a simple interest rate of 6.18% will earn the same amount at the end of one year as a compound interest rate of 6%. ■

Inflation

Inflation, as we see it in the economy, is the periodic rise in our cost of living. As money loses value over time, inflation rises. The variance in inflation depends on the interest reflected in the amounts borrowed or lent. The inflation rate is usually expressed in monthly or annual price increases and published by the Bureau of Labor Statistics in the United States.

This inflation rate is called the **consumer price index (CPI)**. The CPI reflects the prices of certain goods purchased by many people. The values are averages, or calculated means for that year. The percent change shows the change in the CPI from the previous year.

Table 5.1: Consumer Price Index (CPI)

Year	Average CPI	Percent Change in CPI
1985	107.6	3.6
1986	109.6	1.9
1987	113.6	3.6
1988	118.3	4.1
1989	124.0	4.8
1990	130.7	5.4
1991	136.2	4.2
1992	140.3	3.0
1993	144.5	3.0
1994	148.2	2.6
1995	152.4	2.8
1996	156.9	3.0
1997	160.5	2.3
1998	163.0	1.6
1999	166.6	2.2
2000	172.2	3.4
2001	177.1	2.8
2002	179.9	1.6
2003	184.0	2.3
2004	188.9	2.7
2005	195.3	3.4
2006	201.6	3.2
2007	207.3	2.8

Besides inflation rates, the CPI can also record **deflation**, or when the price level decreases from the previous year. However, severe deflation has not occurred since the 1930s. Minor deflations in the economy, called **recessions**, occur when the general economy slows down for a brief period of time.

Inflation usually fluctuates gradually over time. Because of this, we can estimate the inflation rate by using the continuous compounding formula,

$$A = Pe^{rt}$$

Example 5.12. Joe makes a salary of $\$32,000$ per year. About what salary would he need 20 years form now in order to maintain his purchasing power if there was a 4% inflation rate?

Solution. We can use the continuous compounding formula to compute the answer, where $P = \$32,000,\ r = 0.04$, and $t = 20$.

$$A = (\$32,000)(e)^{0.04\times 20} = \$71,217.31.$$

Joe would need to make $\$71,217.31$ per year to maintain his purchasing power. ■

You can also compare a given consumer product with the average inflation. This is called the **inflation proportion**

$$\frac{\text{price in year A}}{\text{price in year B}} = \frac{\text{CPI in year A}}{\text{CPI in year B}}$$

in which CPI is obtained from Table 5.1.

Example 5.13. Rachael's college tuition in 2007 was $\$11,460$. Her aunt attended the same college in 1987 and paid $\$4,120$ in tuition. Compare the college's tuition increase to average inflation over the same period.

Solution. To compare tuition and inflation we will use the inflation proportion formula. First we will find the expected tuition for 2007 by letting x represent the expected value. Then we will compare the actual versus the expected.

$$\begin{aligned}
\frac{\text{price in year A}}{\text{price in year B}} &= \frac{\text{CPI in year A}}{\text{CPI in year B}} \\
\frac{x}{\$4,120} &= \frac{207.3}{113.6} \\
x &= \frac{207.3}{113.6} \times \$4,120 \\
x &\approx \$7,518.27
\end{aligned}$$

Now compare the actual 2007 tuition, $\$11,460$, with the expected figure, $\$7,518.27$.

$$\frac{\$11,460}{\$7,518.27} = 1.52$$

Between 1987 and 2007, this college's tuition has increased approximately 52% more than the average CPI rate. ■

Exercises

In Exercises 1-3 you are given the principal, P, the rate, r, and the time, t. With this given information (a) find the simple interest owed and (b) find the future value of the borrowed money.

1. $P = \$7,000,\ r = 5\%,\ t = 3$ years.
2. $P = \$3,250,\ r = 9\%,\ t = 18$ months.
3. $P = \$11,400,\ r = 6\%,\ t = 60$ days.

In Exercises 4-6 use simple interest to find the unknown value using the future value formula.

4. $A = \$6,000,\ r = 3.5\%,\ t = 4$ years.

5. $A = \$2,700,\ P = \$2,000,\ t = 10$ months.
6. $A = \$5,000,\ P = \$3,500,\ r = 4\%$.

In Exercises 7-13 use simple interest to find the unknown value.

7. Emma Klien bought a new supply of delivery flowers. She paid $\$750$ for the flowers and agreed to pay for them in 6 months at a 7% simple interest. Find the amount of interest that she will owe.
8. Christopher Heart took out an 8% simple interest loan today which will be repaid 10 months from now with a payoff amount of $\$924.75$. What amount is Christopher borrowing?
9. You borrow $\$1,750$ from your best friend and promise to pay him/her back $\$2,100$ in 1 year. What simple interest rate will you have to pay?
10. A bank offers a CD that pays a simple interest rate of 4.5%. How much must you put in this CD now in order to have $\$9,000$ in three years?
11. The Internal Revenue Service owes Martha Fried $\$300$ for overpayment of last year's taxes. The IRS will repay the amount at 5% simple interest. Find the total amount Fried will receive if the interest is paid for 10 months.
12. What is the maximum amount you can borrow today if it must be repaid in 6 months with simple interest at 7% and you know that at that time you will be able to repay no more than $\$1,200$?
13. In order to start a small business, Joe takes out a simple interest loan for $\$5,000$ for 12 months at a rate of 8.5%.
    (a) How much interest must Joe pay?
    (b) Find the future value of the loan.

In Exercises 14-18, use compound interest to find (a) the future value and (b) interest earned.

Principal	Rate	Compounded	Time
14. $\$12,000$	4%	annually	1 year
15. $\$7,500$	5%	semiannually	1 year
16. $\$5,000$	3%	quarterly	4 years
17. $\$3,000$	7%	monthly	8 years
18. $\$11,000$	5.6%	daily	7 years

In Exercises 19-22, Tom and Julie want to establish a savings account that will help pay for medical expenses twenty years from now. For each interest rate find the lump sum they must deposit today so that $\$60,000$ will be available at the time of the medical expense.

19. 9% compounded monthly
20. 7.5% compounded monthly
21. 9% compounded quarterly
22. 7.5% compounded quarterly

In Exercises 23-26, use $A = P\left(1 + \frac{r}{n}\right)^{nt}$ *and* $A = Pe^{rt}$ *to solve.*

23. You have $\$6,000$ to invest. Which investment will yield a greater return after 5 years: 7.5% compounded monthly or 7.25% continuous compounding?

24. You have $\$20,000$ to invest. Which investment will yield a greater return after 10 years: 8.25% compounded daily or 8% continuous compounding?

25. Find the accumulated value of an investment of $\$15,000$ for 8 years at an interest rate of 6.5% if the money is (a) compounded semiannually or (b) continuously compounded. Which investment gives a greater return?

26. Find the accumulated value of an investment of $\$12,450$ for 12 years at an interest rate of 10% if the money is (a) continuously compounded, or (b) compounded monthly. Which investment gives a greater return?

In Exercises 27-31, suppose a savings and loan pays a nominal rate of 4.5% *on savings deposits. Find the effective annual yield if interest is compounded as stated.*

27. quarterly
28. annually
29. monthly
30. semiannually
31. daily

In Exercises 32-34, compare bank accounts.

32. Bank A pays a nominal rate of 3.6% compounded daily on deposits. Bank B produces the same annual yield as the first but compounds interest only quarterly. What nominal rate does Bank B pay?

33. After what time period would the interest earned equal the original principal if the account paid 7% interest and compounded daily?

34. How long would it take to double your money in an account paying 7.5% compounded monthly?

In Exercises 35-39, find the estimated future prices for the given annual inflations. (See Table 5.1 for CPI value).

	Year/Inflation				
	2007	2010	2020	2030	2040
Item		2%	3%	5%	10%
35. House	$\$365,000$				
36. Sit down meal	$\$13.00$				
37. Gallon of Gas	$\$3.50$				
38. Toyota Corolla	$\$16,400$				
39. Movie ticket	$\$8.25$				

In Exercises 40-43 use compound interest to solve.

40. At the time of her son's birth, a mother deposits $\$5,000$ in an account that pays 10% compounded monthly interest. What will be the value of the account at the son's eighteenth birthday, assuming that no other deposits or withdrawals have been made?

41. Joanna wants to have $\$60,000$ available for her college education. If she has 10 more years until college, how much money must she invest at an 8% rate, compounded semiannually to meet her financial goal?

42. A 35-year-old construction worker plans to retire at the age of 60. He believes that $\$450,000$ is needed to retire comfortably. How much should be deposited now at 8.5% compounded annually to meet the $\$450,000$ retirement goal?

43. Max went to Concordia College in 1987 for $\$9,500$; in 2004, his son attended Concordia for $\$22,460$. Compare the school's tuition increase to the average inflation over the same period. (See Table 5.1 for CPI value).

5.2 Loans

In the last section, we looked at how simple and compound interest affected lending and borrowing. We also saw that most of the examples and practice problems dealt with lending or depositing money into a bank. Now we will discuss the borrowing aspect of banks and credit cards.

Closed-end Loans

There are two types of credit. The first type is called **closed-end**, also referred to as **fixed installment**. This **closed-end credit** is used when borrowing an up front set amount and paying that amount off in a series of equal installments (payments). Some examples include purchasing an automobile, furniture, a house, or any other large purchase you cannot pay up front.

As an installment loan, closed-end credit is often based on **add-on interest**. In other words, if an amount P, the principal, is borrowed, then the interest rate, r, and the number of years, t, it takes until repayment are combined into a simple interest formula to compute the future amount of the loan. This formula is the future value formula,

$$A = P(1 + rt)$$

You must also remember that the principal is the amount needed after the **down payment**, the amount due at the loan signing.

Example 5.14. Hunter and Mindy are a newlywed couple. They buy $\$4,500$ worth of furniture and appliances to furnish their first apartment. They pay $\$800$ down and agree to pay the balance at a 6.5% add-on rate for 4 years. Find

(a) the amount financed,

(b) the total amount to be repaid,

(c) the monthly payment,

(d) the total installment price (the sum of the down payment and the total amount to be repaid), and

(e) the finance charge.

Solution. (a)

$$\begin{aligned}\text{amount financed} &= \text{cash price} - \text{down payment}\\ &= \$4,500 - \$800\\ &= \$3,700\end{aligned}$$

(b)

$$\begin{aligned}\text{future amount} &= P(1+rt)\\ &= \$3,700(1+0.065\times 4)\\ &= \$4,662\end{aligned}$$

(c)

$$\begin{aligned}\text{monthly payment} &= \frac{\text{future amount}}{\text{total \# of months}}\\ &= \frac{\$4,662}{48}\\ &= \$97.13\end{aligned}$$

(d)

$$\begin{aligned}\text{total installment price} &= \text{loan repayment} + \text{down payment}\\ &= \$4,662 + \$800\\ &= \$5,462\end{aligned}$$

(e)

$$\begin{aligned}\text{finance charge} &= Prt\\ &= (\$3,700)(0.065)(4)\\ &= \$962\end{aligned}$$

So Hunter and Mindy will end up paying $\$5,462$, which is 30.3% more than the total price of their purchase. ■

Open-end Loans

The second type of credit is **open-end**, also called **installment buying**. Usually when people hear **open-end credit** they think of credit cards. Today, many retail stores offer them with different interest rates and promotions. The credit card is a great example of open-end credit and can also be referred to as **revolving credit**.

Having no planned payment schedule, the open-end loan differs from a scheduled closed-end loan. Credit card companies only require the holder to make minimum monthly payments on their unpaid balance and the interest rate. Compared to other loans, credit cards have high interest rates. However, like other loans, interest is still computed using the simple interest formula $I = Prt$.

Credit card customers are usually billed every month by way of an *itemized bill* that contains the unpaid balance from the first of the period, the total balance owed, a list of purchases and cash advances, the payment due, the minimum payment required, and the date of the last billing period. Besides the itemized bill a customer may have additional **finance charges**. These charges can include an annual fee, interest on unpaid balances, insurance coverage, time payment differential, or carrying charges.

If a customer has an unpaid balance, the credit companies will use the **previous balance method** to calculate the interest on the unpaid balance on the first day of the billing period.

Example 5.15. The table below shows Tommy's MasterCard account activity for a 2 month period. If the bank charges interest of 1.4% per month on the unpaid balance, and there are no other finance charges, find the missing quantities in the table, and the total finance charges for the 2 months.

Month	Unpaid Balance at Beginning of Month	Finance Charge	Purchases During Month	Returns	Payment	Unpaid Balance at End of Month
May	$\$195.67$		$\$98.47$	$\$0$	$\$40$	
June			$\$26.79$	$\$50.98$	$\$90$	

Solution. The May finance charge is 1.4% of $\$195.67$, which is $\$2.74$. Then, adding any finance charge and purchases and subtracting any returns and payments, we arrive at the unpaid balance at the end of May:

$$\$195.67 + \$2.74 + \$98.47 - \$40 = \$256.88$$

A similar calculation gives a June finance charge of

$$0.014 \times \$256.88 = \$3.60$$

and an unpaid balance at the end of June of

$$\$256.88 + \$3.60 + \$26.79 - \$50.98 - \$90 = \$146.29.$$

The total finance charges for the 2 months were

$$\$2.74 + \$3.60 = \$6.34$$

■

Besides the previous balance method, banks will also use the **unpaid balance method**, or the **average daily balance method**. Let's compare these three methods!

Example 5.16. A credit card has a monthly rate of 1.8%. In the March 1-March 31 itemized billing, the March 1 unpaid balance is $\$1,900$. A payment of $\$1,000$ was received on March 8. There are no purchases or cash advances in this billing period. The payment due date is April 9. Find the interest due on this date using each of the three methods for calculating credit card interest.

Solution. We know the monthly rate is 1.8% for all three methods, therefore we will use $I = Prt = P \times 0.018 \times 1$ month to find the interest.

(a) The Unpaid Balance Method: The principal, P, is the unpaid balance on the first day of the billing period less any payments or credits. Then we find the interest on that

$$I = (\$1,900 - \$1,000)(0.018)(1) = \$16.20$$

The interest due on the payment due date is $\$16.20$.

(b) The Previous Balance Method: The principal, P, is the unpaid balance on the first day of the billing period, $\$1,900$. The interest is

$$I = (\$1,900)(0.018)(1) = \$34.20$$

The interest due on the payment due date is $\$34.20$.

(c) The Average Daily Balance Method: The principal, P, is the average daily balance. Next add the unpaid balances for each day in the billing period and divide by the number of days in the billing period, 31. The unpaid balance on the first day of the billing period is $\$1,900$ and a $\$1,000$ payment is recorded on March 8. The sum of the balances owed for each day of the billing period is $\$1,900$ added for each of the first 7 days, plus $\$900$ added for each of the remaining 24 days of the month.

$$\begin{aligned}\text{average daily balance} &= \frac{\text{sum of unpaid balances during the billing period}}{\text{number of days in the billing period}}\\ &= \frac{\$1,900(7) + \$900(24)}{31}\\ &\approx \$1,125.81\end{aligned}$$

The average daily balance is the principal. Therefore the interest is

$$I = (\$1,125.81)(0.018)(1) \approx \$20.27$$

The interest due on the payment due date is $\$20.27$. ■

APR

To find your true annual interest rate, the Consumer Credit Protection Act, commonly known as the **Truth in Lending Act**, was passed in 1968 and addressed many credit issues including:

- How can I tell the true annual interest rate a lender is charging?
- How much of the finance charge am I entitled to save if I decide to pay off a loan sooner than originally scheduled?

Because of the Truth in Lending Act, all sellers must disclose the Annual Percentage Rate (APR), also called the true annual interest rate, stated in contract. By disclosing the APR it lets the borrower compare the true costs of different loans.

When comparing loans you need to follow these steps to find the best loan for you.

STEP 1: Calculate the finance charge (per $\$100$) of the amount financed:

$$\frac{\text{finance charge}}{\text{amount financed}} \times 100$$

STEP 2: Look at the APR table (Table 5.2), find the column corresponding to the number of payments to be made and find the entry closest to the value in step 1.

STEP 3: Find the APR at the left of the row in which the entry from step 2 is found.

Example 5.17. Recall that Hunter and Mindy paid $\$800$ down on a $\$4,500$ purchase and agreed to pay the balance at a 6.5% add-on rate for 4 years. Find the APR for their loan.

Solution. As shown previously, the total amount financed was

$$\text{purchase price} - \text{down payment} = \$4,500 - \$800 = \$3,700$$

The finance charge (interest) was

$$I = (\$3,700)(0.065)(4) = \$962$$

STEP 1: Find the finance charge per $\$100$ of the amount financed. First divide the finance charge by the amount financed, then multiply by $\$100$.

$$\begin{aligned} \text{finance charge per \$100} &= \frac{\text{finance charge}}{\text{amount financed}} \times \$100 \\ &= \frac{\$962}{\$3,700} \times \$100 \\ &= \$26 \end{aligned}$$

Table 5.2: APR table

Annual Percentage Rate (APR)	Number of Monthly Payments							
	6	12	18	24	30	36	48	60
8.0%	\$2.35	4.39	6.45	8.55	10.66	12.81	17.18	21.66
8.5%	\$2.49	4.66	6.86	9.09	11.35	13.64	18.31	23.10
9.0%	\$2.64	4.94	7.28	9.64	12.04	14.48	19.45	24.55
9.5%	\$2.79	5.22	7.69	10.19	12.74	15.32	20.59	26.01
10.0%	\$2.94	5.50	8.10	10.75	13.43	16.16	21.74	27.48
10.5%	\$3.08	5.78	8.52	11.30	14.13	17.01	22.90	28.96
11.0%	\$3.23	6.06	8.93	11.86	14.83	17.86	24.06	30.45
11.5%	\$3.38	6.34	9.35	12.42	15.54	18.71	25.23	31.96
12.0%	\$3.53	6.62	9.77	12.98	16.24	19.57	26.40	33.47
12.5%	\$3.68	6.90	10.19	13.54	16.95	20.43	27.58	34.99
13.0%	\$3.83	7.18	10.61	14.10	17.66	21.30	28.77	36.52
13.5%	\$3.97	7.46	11.03	14.66	18.38	22.17	29.97	38.06
14.0%	\$4.12	7.74	11.45	15.23	19.10	23.04	31.17	39.61
14.5%	\$4.27	8.03	11.87	15.80	19.81	23.92	32.37	41.17
15.0%	\$4.42	8.31	12.29	16.37	20.54	24.80	33.59	42.74
15.5%	\$4.57	8.59	12.72	16.94	21.26	25.68	34.81	44.32
16.0%	\$4.72	8.88	13.14	17.51	21.99	26.57	36.03	45.91

STEP 2: To find their APR look at the column stating 48 monthly payments. Then look down the table until you see the finance charge closest to $\$26$.

STEP 3: Read across the row to find the APR value. Thus the APR value for their loan is 12% ■.

Exercises

In Exercises 1-5, use closed-end credit to solve the problems.

1. Suppose you want to buy a new car that costs $\$21,400$. You have no cash – only your old car, which is worth $\$4,000$ as a trade-in. The new car has an interest rate of 8% add-on for 4 years.

   (a) Find the amount financed.

   (b) Find the total interest.

   (c) Find the total amount to be repaid.

   (d) Find the monthly payments.

   (e) Find the total installment price.

2. The cash price for furniture for all rooms of a two bedroom house is $\$10,450$. The furniture can be financed by paying $\$600$ down with an interest rate of 5% add-on for 2 years.
   (a) Find the amount financed.
   (b) Find the total interest.
   (c) Find the total amount to be repaid.
   (d) Find the monthly payments.
   (e) Find the total installment price.
3. The cost of a Macbook is $\$1,299$. We can finance this by paying $\$600$ down and $\$40$ per month for 24 months. Determine
   (a) the amount financed.
   (b) the total installment price.
   (c) the finance charge.
4. The cost of a washer-dryer is $\$2,700$. We can finance this by paying $\$300$ down and $\$200$ per month for one year. Determine
   (a) the amount financed.
   (b) the total installment price.
   (c) the finance charge.
5. You plan to buy a home entertainment system for $\$1,400$. The store will approve your loan if you put $\$1,000$ down and agree to pay $\$30$ per month for 18 months.
   (a) Find the amount financed.
   (c) Find the total installment price.
   (d) Find the finance charge.

In Exercises 6-9, find (a) the finance charge, and (b) the APR.

6. Robby Lin financed a $\$2,310$ computer with 28 monthly payments of $\$102.50$ each.
7. Patsy Paterson bought a jet ski for $\$3,080$. She paid $\$700$ down and paid the remainder at $\$206$ per month for 2 years.
8. Mikal Kiss still owed $\$3,000$ on his new garden tractor after the down payment. He agreed to pay monthly payments for 18 months at 7% add-on interest.
9. Phil Divan paid off a $\$16,000$ car loan over 4 years with monthly payments of $\$485.45$ each.

In Exercises 10-14, use open-ended credit to solve the problems.

10. A particular MasterCard calculates interest using the unpaid balance method. The monthly interest rate is 1.6% on the unpaid balance on the first day of the billing period less payments and credits. Here are some of the details in the September 1 - September 30 itemized billing:

    September 1 Unpaid Balance: $\$870$; Payment Received September 8: $\$100$; Purchases Charged to the MASTER Account: clothing $\$70$, and gasoline $\$67$; Last Day of the Billing Period: September 30.

    (a) Find the interest due on the last day of the billing period.

    (b) Find the total balance owed on the last day of the billing period.

    (c) The MasterCard requires a $\$25$ minimum monthly payment if the total balance owed on the last day of the billing period is less than $\$400$. Otherwise, the minimum monthly payment is $\frac{1}{20}$ of the balance owed on the last day of the billing period, to the nearest whole dollar. What is the minimum monthly payment due for this billing period?

11. A Visa card calculates interest using the unpaid balance method. The monthly interest rate is 17.6% on the unpaid balance on the first day of the billing period less payments and credits. Here are some of the details in the December 1 - December 31 itemized billing:

    December 1 Unpaid Balance: $\$549$; Payment Received December 8: $\$300$; Purchases Charged to the Account: food $\$110$, gasoline $\$48$, other $\$170$; Last Day of the Billing Period: December 31.

    (a) Find the interest due on the last day of the billing period.

    (b) Find the total balance owed on the last day of the billing period.

12. A Sears card has a monthly rate of 3.5% and uses the average daily balance method for calculating interest. Here are some of the details in the May 1 - May 31 itemized billing:

    May 1 Unpaid Balance: $\$3,999$; Payment Received May 8: $\$250$; Purchases Charged to the Sears Account: $\$0$; Last Day of the Billing Period: May 31.

    (a) Find the interest due on the last day of the billing period.

    (b) Find the total balance owed on the last day of the billing period.

13. A credit card has a monthly rate of 7.4%. In the July 1-July 31 itemized billing, the July 1 unpaid balance is $\$1,000$. A payment of $\$350$ was received on July 6. There are no purchases or cash advances in this billing period. Find the interest due for this billing period using:

    (a) the unpaid balance method.

    (b) the previous balance method.

(c) the average daily balance method.

14. A credit card has a monthly rate of 2.2%. In the March 1 - March 30 itemized billing, the March 1 unpaid balance is $\$190$. A payment of $\$50$ was received on March 5. There are no purchases or cash advances in this billing period. Find the interest due for this billing period using:

    (a) the unpaid balance method.

    (b) the previous balance method.

    (c) the average daily balance method.

15. Which method for calculating credit card interest is most beneficial to the borrower and which is least beneficial if no charges or credits are recorded in a billing cycle, only payments? Explain why this is so.

16. A Sears charge card has a monthly rate of 1.5%. The interest is a minimum of $\$0.80$ if the average daily balance is $\$25.00$ or less. Explain how this policy is beneficial to Sears.

5.3 Mortgages

Purchasing and owning a home is one of the largest financial commitments in a person's life and is part of the American dream. When purchasing a home, many factors must be considered that go beyond the initial price of the home. A buyer must also take into account their annual gross income and their ability to pay the monthly payments that could persist for 30 years. In this section, we will cover affordability principles, the basic concepts of a home mortgage, different types of mortgages, and a down payment.

Affordability Principles

When purchasing a home, a buyer's ability to afford a house depends on their income and debts; although, there are a few principles that can be used to approximate how much a buyer is able to spend on a new home. The following are two of the most widely used:

1. Assuming a typical down payment of 20% of the price of the home, the maximum home price should not be more than 3 times the buyer's annual gross income, or income before deductions. That is,

$$\text{maximum home price} = 3 \times (\text{annual gross income})$$

2. The monthly housing expenses, including the mortgage payment (which will be described later in the section), homeowner's insurance, and property taxes should not exceed 25% of the buyer's monthly gross income. Monthly gross

income can be calculated by taking $\frac{1}{12}$ of the annual gross income. Thus, we have

$$\text{maximum monthly housing expenses} = 0.25 \times \frac{1}{12} \times (\text{annual gross income}).$$

These guidelines define the *maximum* affordable home price and monthly housing expenses. It is very common that a buyer finds a home that consists of a lower price and, thus, lower housing expenses. Our second principle states that the maximum monthly housing expenses should be less than 25% of the buyer's monthly gross income; however, some banks allow these monthly housing expenses to be up to 38% of the buyer's monthly gross income. Thus, we will call the 25% amount the *low maximum monthly housing expense estimate* and the 38% amount the *high maximum monthly housing expense estimate*.

Example 5.18. Suppose that your family is looking to buy a new home and your annual gross income is $\$90,000$. According to the affordability principles, what are the maximum home price and monthly housing expenses for your potential home?

Solution. A potential home price for your family can not exceed three times your family's annual gross income. Thus,

$$\text{maximum home price} = 3 \times \$90,000 = \$270,000$$

The monthly housing expenses can not exceed 25% of your family's monthly gross income. Thus,

$$\text{maximum monthly housing expenses} = 0.25 \times \frac{1}{12} \times \$90,000 = \$1,875.$$

■

If we had not specified to calculate the maximum monthly housing expenses *according to the affordability principles* you could have calculated both the low maximum monthly housing expense estimate at the 25% level, which we did above, and the high maximum monthly housing expense estimate at the 38% level, which is the following:

$$\text{high maximum monthly housing expense estimate} = 0.38 \times \frac{1}{12} \times \$90,000 = \$2,850.$$

This gives us two potential estimates of how high a buyer's maximum monthly housing expenses can be. We are now going to cover the different types of mortgages, how to calculate your mortgage, and determining whether you qualify for a particular mortgage.

The Down Payment

After a buyer finds a home that they can afford, they will most likely need to seek out a mortgage from a bank. A **mortgage** is a long-term loan in which the property is guaranteed as security for the payment of the difference between the sale price and the down payment. The **down payment** is the amount of cash that the buyer must pay before the bank grants them a mortgage and can vary from 5% to 50% of the price of the home. For example, a buyer of a house with a cost of $\$150,000$ may require a down payment of $\$30,000$. (That is, $0.2 \times \$150,000 = \$30,000$). With a down payment of 20% of the property value, the affordable maximum home price can also be determined with this information as shown in the following example.

Example 5.19. What is the highest maximum home price with a 20% down payment that you can afford if you have $\$30,000$ for the down payment?

Solution. The down payment is 20% of the maximum home price that you can afford to pay. We will let x denote the price of the home so 20% of x is $\$30,000$, or $0.20x = \$30,000$. Therefore,

$$x = \frac{\$30,000}{0.20} = \$150,000$$

Thus, the maximum home price that you can afford is $\$150,000$. ■

This gives us the formula,

$$\text{Maximum Affordable Home Price} = \frac{\text{Down payment dollars}}{\text{Down payment percentage}}$$

The size of the down payment is dependent on the bank or institution from which you are receiving the money, the age of the property, whether money is "tight", or difficult to borrow at that time, and the type of mortgage loan. Typically, older homes require larger down payments and newer homes require smaller down payments.

Example 5.20. Suppose you are looking at a house with a price of $\$175,000$ that requires a down payment of 20%. Calculate the required down payment amount.

Solution. Let x represent the required down payment amount, which is 20% of $\$175,000$. Then we have,

$$x = 0.20 \times \$175,000 = \$35,000.$$

Thus, the house requires a down payment of $\$35,000$. ■

Types and Characteristics of Mortgages

Although there are many different types of mortgages, the two most common are the **conventional loan**, also referred to as a **fixed-rate mortgage** and the **adjustable-rate mortgage**, also called a **variable-rate loan**. We will use the terms fixed-rate mortgage (FRM) and adjustable-rate mortgage (ARM). The major difference between these two types of mortgages is that the ARM has a fixed interest rate only for the first few years and is later modified up or down depending on the interest rates at the time, whereas the FRM receives an interest rate at the time the loan is made that remains constant for the duration of the loan.

So how do you know if you should want a fixed-rate mortgage or an adjustable-rate mortgage? The answer to that question depends on the interest rate at the time you receive your loan and the predicted interest rate for the future. If you receive your loan at a time when the interest rate is low and is expected to rise, it would be more practical to obtain a FRM. That way, when the interest rate rises you will continue to pay your mortgage based on the initial lower interest rate. In contrast, if interest rates are intended to go down during the life of your mortgage, it would be best to obtain a ARM that would allow for its interest rate to also decrease. Sadly, it is very difficult to predict the future behavior of the interest rates and cannot be done with assurance.

Another important characteristic of a mortgage is its **term**, or the amount of time over which the loan will be repaid. The most typical term choices are 10, 15, or 30 years. The longer the term, the smaller the monthly payment and vice versa. Also, the longer term usually has a higher interest rate since the money is being borrowed for a longer period of time.

Points and Closing Costs

The bank or institution from which money is being borrowed sometimes requires the buyer to pay one or more *points* toward a loan at the time of **closing**, the final step toward purchasing the home. A **point** is interest paid upfront to the lender that can reduce the interest rate. One point is equal to 1% of the total principal of a loan. There are two types of points that the lender could charge:

- **Loan origination fee**: a fee for taking out a loan
- **Discount charge**: a fee for receiving a lower interest rate

The loan origination fee and the discount charge make up the **closing costs** that are paid at the time of closing.

Example 5.21. Suppose you take out a loan of $\$100,000$ at 6% interest on a 30-year fixed-rate mortgage. The loan you will be taking out consists of a 1-point discount charge and a 2-point loan origination fee. What will your additional costs add up to?

Solution. Since each point is equal to 1% of the amount of the loan, the discount charge will cost 1% of the loan amount and the loan origination fee will cost 2% of the loan amount. Thus, the discount charge is

$$0.01 \times \$100,000 = \$1,000$$

and the loan origination fee is

$$0.02 \times \$100,000 = \$2,000.$$

The total amount of extra charges will equal $\$1,000 + \$2,000 = \$3,000$. ■

Example 5.22. The Olson's would like to purchase a new home that is selling for $\$319,000$. They are planning on acquiring a loan from their bank. The bank requires a down payment of 20% of the price of the home, payable to the seller, and a payment of 3 points, payable to the lender, at the time of closing.

(a) Calculate the down payment the Olson's are required to pay.

(b) Calculate the Olson's mortgage loan amount.

(c) Calculate the cost of the 3 points on the Olson's mortgage.

Solution. (a) The required down payment is 20% of $\$319,000$, which is

$$0.20 \times \$319,000 = \$63,800$$

(b) The mortgage is equal to the difference between the selling price and the down payment, or

$$\$319,000 - \$63,800 = \$255,200$$

(c) The cost of three points is equal to 3% of the loan amount, or

$$0.03 \times \$255,200 = \$7,656$$

Thus, at the closing, the Olson's will be required to pay the down payment of $\$63,800$ to the seller and $\$7656$ to the bank. ■

Qualifying for a Mortgage

In order to determine whether a buyer qualifies for a mortgage, the bank uses a formula to determine the monthly payment amount that they believe the borrower will be able to afford.

Let m represent the buyer's **adjusted monthly income**, which is determined by subtracting any unchanging monthly payments that still have more than 10 months remaining (i.e. student loans, car payments) from their gross monthly income. The formula used is as follows:

$$0.28m = p$$

where p is the maximum payment amount that the bank believes the buyer can afford each month. This monthly payment must include property taxes, interest, homeowners' insurance, and principal.

Example 5.23. Suppose the Peterson's went to the bank to receive a loan for the purchase of a new home. They have a gross monthly income of $\$6,725$. They have 12 remaining monthly payments of $\$125$ on their student loans and 24 remaining monthly payments of $\$275$ on their car loan. Determine the monthly payment amount that the bank would believe the Peterson's would be able to afford.

Solution. The Peterson's adjusted monthly income equals

$$\$6,725 - \$125 - \$275 = \$6,325$$

According to the above formula, they are able to make monthly payments of

$$0.28 \times \$6,325 = \$1,771$$

■

Example 5.24. Suppose the Johnsons' gross monthly income is $\$8,220$. They have 17 remaining monthly payments of $\$220$ on their car loan. They have 11 remaining monthly payments of $\$310$ on their son's student loans and they have 22 remaining monthly payments of $\$140$ on their credit card bill. The homeowners' insurance and property taxes on their potential home are $\$185$ and $\$135$, respectively. A monthly mortgage payment of property taxes, interest, homeowners' insurance, and principal that is less than or equal to 28% of their adjusted monthly income will be approved by the bank.

(a) Calculate 28% of the Johnsons' adjusted monthly income.

(b) The Johnsons want a 30-year, $\$234,660$ mortgage. Supposing an interest rate of 6.5%, calculate the total monthly mortgage payment (including property taxes, interest, homeowners' insurance, and principal) that would be required with this mortgage.

(c) Decide whether the Johnsons qualify for this particular mortgage or not.

Solution. (a) First we must calculate the Johnsons' adjusted monthly income. To do this, we must calculate the difference between the sum of their monthly payments, $\$220 + \$310 + \$140 = \$670$, and their gross monthly income, $\$8,220$.

$$\$8,220 - \$670 = \$7,550$$

Next, we will take 28% of the adjusted monthly income.

$$0.28 \times \$7,550 = \$2,114$$

Therefore, 28% of the Johnsons' adjusted monthly income is $\$2,114$.

(b) In order to determine the total monthly mortgage payment, we must first determine the monthly principal and interest payments since we were given their homeowners' insurance and property taxes. We will use Table 5.3 to find the monthly principal and interest payments that the Johnsons will have to pay. According to the table, with a 30-year mortgage and an interest rate of 6.5%, the Johnsons would be required to pay a monthly principal and interest payment of $\$6.32$ per thousand dollars of mortgage.

In order to determine their monthly principal and interest payment, we will first have to divide the mortgage amount by $\$1,000$.

$$\frac{\$234,660}{\$1,000} = 234.66$$

Table 5.3: Monthly Principal and Interest Payment per $1000 of Mortgage

Rate %	10-year	20-year	30-year
5.0	10.61	6.60	5.37
5.5	10.85	6.88	5.68
6.0	11.10	7.16	6.00
6.5	11.35	7.46	6.32
7.0	11.61	7.75	6.65
7.5	11.87	8.06	6.99
8.0	12.13	8.36	7.34

This is the number of thousands of dollars of the mortgage.

So in order to determine the monthly principal and interest payment, we have to multiply the number of thousands of dollars of the mortgage, 234.66, by the number that we obtained from Table 5.3.

$$234.66 \times \$6.32 \approx \$1,483.05$$

Therefore, the monthly principal and interest payment is approximately $\$1483.05$.

To calculate the total monthly mortgage payment, we must find the sum of the monthly principal and interest payment, $\$1,483.05$, the cost of homeowners' insurance, $\$185$, and property taxes, $\$135$.

$$\$1,483.05 + \$185 + \$135 = \$1,803.05$$

Thus, the Johnsons have a total monthly mortgage payment of $\$1,803.05$.

(c) In order for the Johnsons to qualify for this mortgage according to lender requirements, the monthly mortgage payments cannot exceed 28% of their adjusted monthly income. In part (a) we calculated 28% of their adjusted monthly income to be $\$2,114$ and in part (b) we calculated the total monthly mortgage payment to be $\$1,803.05$. Since $\$2,114 > \$1,803.05$, the Johnsons would qualify for this particular mortgage. ■

Exercises

For Exercises 1-5 calculate the necessary 20% down payment if the cost of the house is:

1. $\$375,000$
2. $\$124,500$
3. $\$236,000$
4. $\$318,600$
5. $\$386,200$

For Exercises 6-15 calculate the maximum affordable home price and monthly housing expenses for your potential home according to affordability principles if you have the given annual gross income.

6. Annual gross income is $\$95,000$
7. Annual gross income is $\$72,000$
8. Annual gross income is $\$62,000$
9. Annual gross income is $\$110,000$
10. Annual gross income is $\$120,600$
11. Annual gross income is $\$100,000$
12. Annual gross income is $\$145,000$
13. Annual gross income is $\$85,000$
14. Annual gross income is $\$160,000$
15. Annual gross income is $\$147,000$

For Exercises 16-18 calculate the couple's adjusted monthly income.

16. The Martins' gross monthly income is $\$9,720$. They have 12 car payments of $\$155$ a month and they have 16 student loan payments of $\$225$ a month remaining.

17. The Olson's have a gross monthly income of $\$8,200$. They have 13 payments of $\$160$ a month remaining on their credit card bill. They also have 20 payments of $\$280$ a month on their car loan.

18. The Greens' gross monthly income is $\$7,800$. They still have 14 monthly payments of $\$230$ remaining on their credit card bill and 17 monthly payments of $\$140$ remaining on their student loans.

For Exercises 19-24 it is necessary to use Table 5.3.

19. The Bailey's gross monthly income is $\$10,300$. They have 11 monthly payments of $\$165$ remaining on their student loans and 14 monthly payments of $\$265$ remaining on their credit card bill. Calculate 28% of the Bailey's adjusted monthly income.

20. Refer to problem 19. The Bailey's want a 30-year, $\$280,000$ mortgage. Supposing an interest rate of 6.5%, calculate the total monthly mortgage payment that would be required with this mortgage.

21. Refer to problems 19 and 20. Do the Bailey's qualify for this particular mortgage. Why or why not?

22. The Robson's have a gross monthly income of $\$8,720$. They have 16 payments of $\$170$ remaining on their car loan. Calculate 28% of the Robson's adjusted monthly income.

23. Refer to problem 22. The Robson's want a 30-year, $\$195,000$ mortgage. Supposing an interest rate of 6.0%, calculate the total monthly mortgage payment that would be required with this mortgage.

24. Refer to problems 22 and 23. Do the Robson's qualify for this particular mortgage. Why or why not?

For Exercises 25-27 refer to the following situation: The Jones' are wanting to purchase a new home with a price of $\$295,000$*. They have gone to their bank to obtain a loan and the bank is requiring a down payment of* 20% *of the price of the home, paid to the seller, and a payment of* 2 *points, paid to the lender, that are paid at the time of closing.*

25. Calculate the down payment required of the Jones'.
26. Calculate the Jones' mortgage amount.
27. Calculate the cost of the 2 points on the Jones' mortgage that is payable to the bank.

6

Statistics

For centuries, statistics has played an important role in government, advertisements/marketing, public opinion, and comparative analysis. Statistics help us see the natural or unnatural changes during the course of history. By collecting, organizing, analyzing, interpreting, and drawing conclusions we can look at the present readings and resume or change our habits based on the conclusions. This method is called **statistics**.

Essentially there are two types of statistics. **Descriptive statistics** is the method of collecting, organizing, summarizing, and presenting the data. Usually a table or a graph will appear with descriptive statistics. The other type is **inferential statistics**. This method is used to make generalizations and draw conclusions from the collected data. Thus, inferential statistics will not take place until after descriptive statistics have been collected and presented.

6.1 Population and Samples

Education is on the minds of many Americans. In Table 6.1, each state shows the set of all students graduating 9th grade and then the set of all high school graduates. In general, these sets, also called the **population**, are the sets that contain all of the people or objects' properties as described and analyzed by the data collector. As you can see, the population of the nation's 9th grade graduates was $4,012,333$, while the number of high school graduates was $2,760,672$ for the 2004-2005 school year. However, these numbers can not be evenly distributed among our 50 states. Instead a *sample* must be taken from each state. A **sample** is a subset or subgroup of the population needed to draw conclusions about the population as a whole. So while the nation's percentage of graduation rate for 2005 was 68.8%, Minnesota's rate was 84.5%.

Table 6.1: Public High School Graduation Rates Research by The National Center for Higher Education Management Systems

State	Cohort Survival Rate	Fall 2001 9th Graders	2004-05 High School Graduates
Alabama	61.4	61,038	37,453
Alaska	58.9	11,734	6,909
Arizona	81.6	72,859	59,436
Arkansas	74.2	35,894	26,621
California	71.1	499,505	355,214
Colorado	71.0	62,756	44,532
Connecticut	76.2	46,621	35,515
Delaware	65.4	10,602	6,934
Florida	53.6	248,764	133,316
Georgia	55.0	128,734	70,834
Hawaii	67.4	16,036	10,813
Idaho	79.1	19,923	15,768
Illinois	74.4	165,529	123,215
Indiana	70.2	78,945	55,444
Iowa	84.3	39,818	33,547
Kansas	78.6	38,621	30,355
Kentucky	71.3	53,583	38,202
Louisiana	63.0	57,164	36,009
Maine	78.3	16,689	13,073
Maryland	73.9	73,300	54,170
Massachusetts	74.2	80,394	59,665
Michigan	69.7	145,651	101,582
Minnesota	84.5	69,032	58,337
Mississippi	61.1	38,498	23,523
Missouri	77.0	75,156	57,841
Montana	79.0	13,004	10,270
Nebraska	83.6	23,855	19,939
Nevada	49.1	32,086	15,740
New Hampshire	78.0	17,646	13,771
New Jersey	87.6	98,784	86,488
New Mexico	60.2	28,816	17,353
New York	62.4	245,540	153,203
North Carolina	65.7	114,236	75,010
North Dakota	84.8	8,906	7,552
Ohio	74.9	155,727	116,695
Oklahoma	73.9	49,034	36,225
Oregon	72.3	45,067	32,602
Pennsylvania	78.0	159,919	124,701
Rhode Island	73.0	13,538	9,881
South Carolina	52.0	64,279	33,439
South Dakota	80.8	10,629	8,584
Tennessee	64.5	74,322	47,947
Texas	65.3	366,895	239,701
Utah	86.4	35,029	30,249
Vermont	83.2	8,595	7,152
Virginia	73.2	100,599	73,667
Washington	70.5	86,396	60,894
West Virginia	73.4	23,328	17,128
Wisconsin	81.3	77,802	63,229
Wyoming	75.5	7,443	5,616
Nation	68.8	4,012,333	2,760,672

Example 6.1. A big-city advertising agency decides to conduct a survey among the citizens of the city to discover their opinion on the most effective form of advertisement.

(a) Describe the population.

(b) One of the agency representatives suggests obtaining a sample by surveying all the people at five of the largest advertising agencies in the city on a Wednesday morning. Each person will be asked to express his or her opinion on the best form of advertisement. Does this seem like a good idea?

Solution. (a) The population is the set containing all the citizens of the city.

(b) Questioning people at five of the city's largest advertising agencies is a terrible idea. The agencies subset is probably more likely to have a partisan attitude toward a particular type of advertising than the population of all the city's citizens. ■

Random Sampling

The graduation data gives a great example of how we can use a small sample to make generalizations about a large population. For instance, since the graduation rate in Minnesota was higher than the national level, we can generalize that the state's average income and education funding are greater than other states. We know Minnesota is a sample of the overall population but is it a *random sample*? In order for the Minnesota data to be a **random sample**, the sample must be obtained in such a way that every citizen in the population has an equal chance of being selected for the sample. One way to obtain a random sample is to identify each type of constituent and assign that person a number. These numbers are then assigned an order, giving each constituent an equal chance.

Frequency Distribution

After the data has been collected, it needs to be presented in an organized manner. This manner is called **frequency distribution**. When a data set includes repeated items, the data can be organized into a table. Each table will have a list of distinct data values (x) and a list of their **frequencies** (f), or the number of times a data value occurred in the data set. The table can also include a **relative frequency**, which is the percentage of data that is represented by the item. To find the relative frequency for each item x, let n be the total number of items that occur f times, then $x = \dfrac{f}{n}$. Thus, each list is set up in column form for a total of three columns.

Example 6.2. Construct a frequency and relative frequency distribution for the following given data of the age of maximum yearly growth for 35 girls.

10	12	11	12	14	12	12	15	11	9
11	16	10	13	12	13	13	14	12	11
13	14	13	10	11	14	13	13	10	12
10	14	11	12	15					

Solution. .

Age	Frequency f	Relative Frequency f/n
9	1	$0.03 = 3\%$
10	5	$0.14 = 14\%$
11	6	$0.17 = 17\%$
12	8	$0.23 = 23\%$
13	7	$0.2 = 20\%$
14	5	$0.14 = 14\%$
15	2	$0.06 = 6\%$
16	1	$0.03 = 3\%$
Total (n)	35	

■

By organizing the data into a frequency distribution, we can easily make sense of the data. First the frequency distribution will be set up as a two column table. The left-hand column includes the data values, from smallest to largest and the right-hand column contains the frequency or number of girls with the same maximum yearly growth. The frequency distribution indicates that one subject has maximum growth at age 9, five at age 10, six at age 11, and so on.

In Example 6.2, notice the sample size was relatively small. In comparison, a larger data set would take longer to process. Therefore, the larger sets are put into groups or *classes*. Putting the data into appropriate classes is called **grouped frequency distribution**. The bottom limit of each class is called the **lower class limit** while the highest limit of each class is called the **upper class limit**. Since each class has a range of numbers, we call the difference between those two limits **class width**. For example, if the lower limit was 30 and the upper limit 39, the class width would be $39 - 30 = 9$.

The next step is to compose classes. And although there are not set rules on how to compose classes, here are a few guidelines to follow.

- Make sure each data value fits into only **one** class,
- Try to make all classes the same width,
- Make sure the classes do not overlap, and
- Use between 5 to 12 classes.

Once the table is finished, a graph can be constructed to display your findings. One type of graph is called a *histogram*. A **histogram** is a series of rectangles that represent the frequencies of each item. The data values are placed on the horizontal axis and the frequencies on the vertical axis.

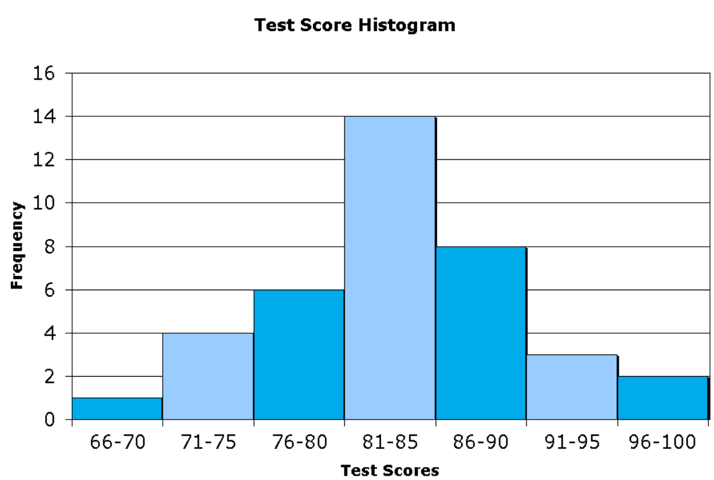
Example 6.3. Use a class width of 4 to construct a grouped frequency distribution and a histogram for the 38 test scores given below.

97	87	82	69	90	92	84	83	76	81
81	84	94	72	73	82	83	89	87	78
80	100	85	85	76	72	95	87	89	81
83	77	76	82	88	87	72	85		

Solution. .

Table 6.2: A frequency distribution for a math class's test scores.

Test Scores	Frequency
$66-70$	1
$71-75$	4
$76-80$	6
$81-85$	14
$86-90$	8
$91-95$	3
$96-100$	2
Total	38



■

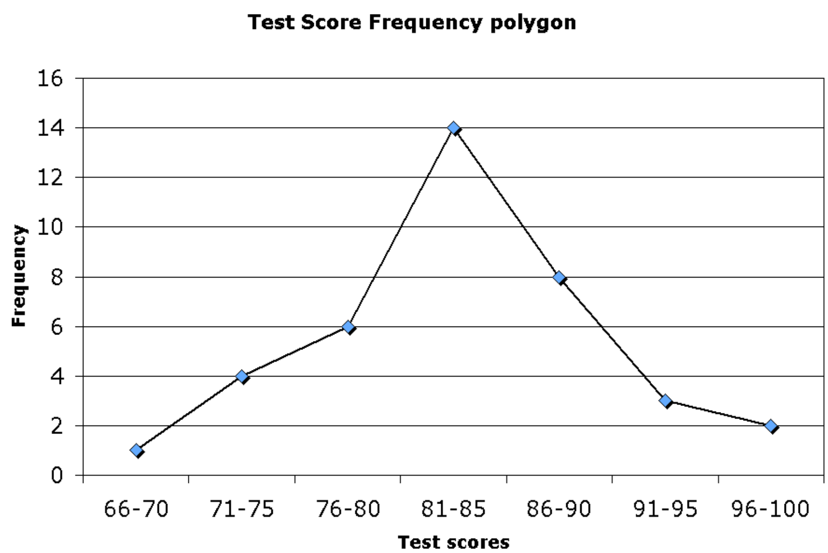


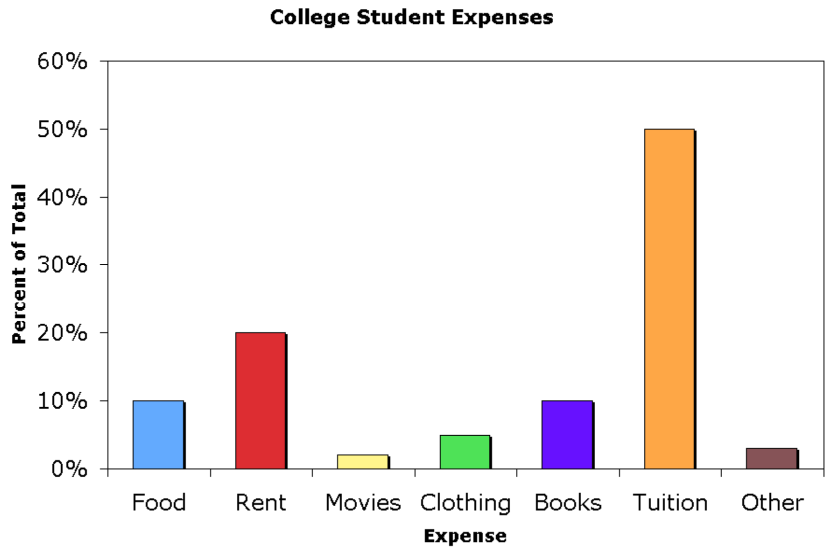
Figure 6.1: A frequency polygon

Another visual graph for frequency distribution is the **frequency polygon**. This visual is a line graph whose points are the ordered pairs of the data values and frequencies. For instance, to find the frequency at the age of 13, which is 5, the ordered pair would be $(13, 5)$. After all of the points are plotted, they are connected with line segments. The frequency polygon can be displayed two ways. The first is combined with the histogram, the second is by itself.

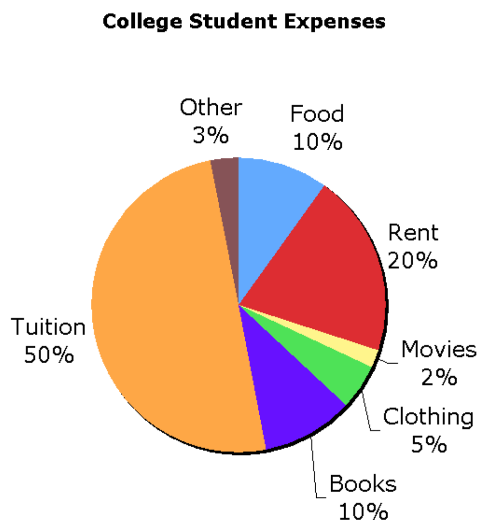
Using the same data as in Example 6.3, the frequency polygon in Figure 6.1 displays a line graph with center points on the midpoint of each class.

Graphs

Typically, when we think of graphs, three come to mind. These three graphs are the bar graph, circle graph, and line graph. Each graph has its own advantages and disadvantages. Now the question is which data goes best with each graph. Let's start with the bar graph. Similar to a histogram, the **bar graph** is composed of rectangles that are not touching. The rectangles can also be arranged vertically or horizontally.



(a) A bar graph



(b) A pie chart

Figure 6.2: Graphs of College Student Expenses

Next, the **circle graph**, or **pie chart**, uses a circle to represent the total number

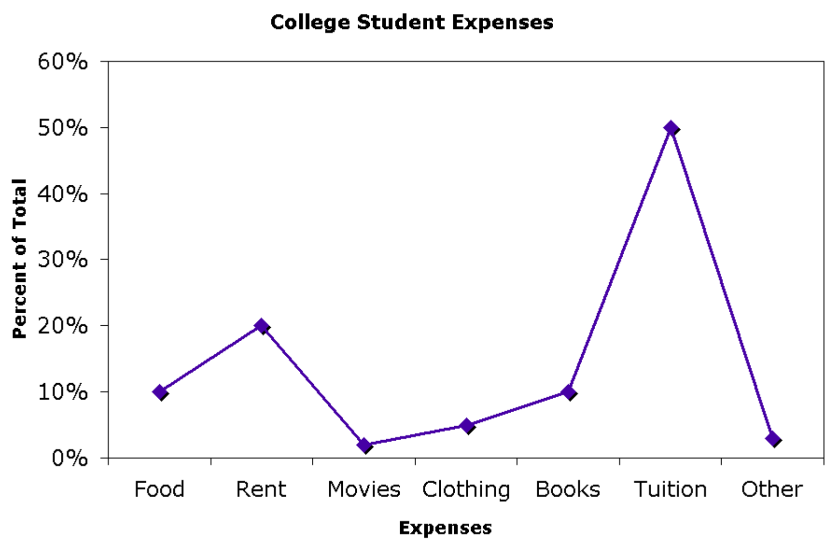


Figure 6.3: A line graph

of categories and is divided into sectors, or wedges, relative to the sizes or magnitudes of the categories. Each sector is a percent of the circle. We know a circle has $360°$. If a sector represents 25% of the circle, then 25% of $360°$ is $(.25)(360) = 90°$.

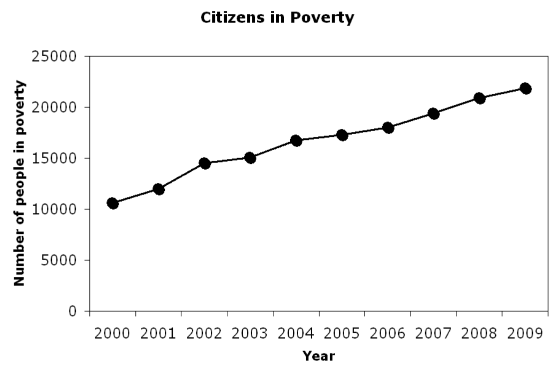
The third graph is frequently used to compare an item's change over time. The **line graph** uses ordered pairs and line segments to compare this change. Different from the other two graphs, the line graph can also plot multiple lines to compare patterns or trends.

The graphs we have looked at so far have been void of deception, but that may not always be the case. Look at the two graphs in Figure 6.4 and determine which better represents the data.

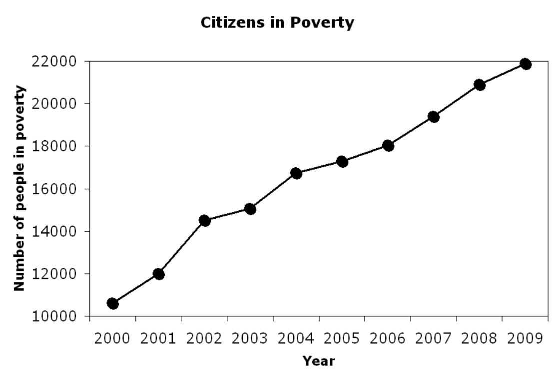
Although poverty is a major issue, it depends on a graph's accuracy to portray the right image. If a local newspaper wanted to spark interest in its readers, it would print the graph in Figure 6.4(b). However, if the newspaper reports the truth, it would select an accurate representation of the data like the graph in Figure 6.4(a). Here are a few pointers to watch for:

1. Does the title explain what is displayed?

2. Is the vertical axis numbered with tick marks that clearly indicate the scale? And has the scale been varied to create a more or less dramatic impression than the actual data?



(a)



(b)

Figure 6.4: Poverty Rates in the United States

3. Are there too many designs and cosmetic effects that draw your attention from or distort the data?

4. Is the horizontal axis spaced equally with the time intervals?

5. Does the scale proportionately represent the data?

6. Is the source of the data and how the data was collected indicated? What is the margin of error in the collected data?

Exercises

1. "The Women Poll" of 1,000 randomly-selected American women ages 18 and older was taken by *Readers Diet* August 10-13, 2006. The readers were asked, "In general, how would you describe your health?"

Rating	Age		
	$18-34$	$35-55$	56
Excellent	31	19	13
Very Good	26	30	35
Good	20	25	31
Fair	22	20	11
Poor	1	6	8
Don't Know	0	0	2

   Answer the following questions for the data shown above.

   (a) Describe the population and the sample of this poll.

   (b) For each woman, what variable is measured? Is the data quantitative(numerical) or qualitative (non-numerical categories)?

   (c) What graph would best fit the data?

2. The American Association of Physical Exercise (May 2001) published the results of a study on the use of stretching before, during, and after a one-hour workout. Of $1,500$ gymnasts who were randomly-selected for the study, 69% stretched before, during, and after their workout.

   (a) Describe the population and the sample of this study.

   (b) Is the sample representative of the population? Explain your answer.

   (c) For each gymnast, what variable is measured? Is the data quantitative (numerical) or qualitative (non-numerical categories)?

   (d) What graph would best fit the data?

A reading teacher had her students keep a record of the number of times they read an article or book during a 5-day week. Excluding test or quiz questions, the article or book in question must take at least 5 minutes to read to be recorded. Shown is the group frequency distribution. Use this data to solve Exercises 3-8.

Reading Rate	Frequency
$0-5$	4
$6-10$	10
$11-15$	39
$16-20$	12
$21-25$	5

3. Which reading rate describes the greatest number of students? How many students responded with this rating?

4. Which reading rate describes the least number of students? How many responded with this rating?

5. How many students were involved in this study?

6. How many students have a reading rate of 16 or more?

7. Construct a histogram.

8. Construct a frequency polygon.

In Exercises 9-11, use Microsoft Excel and the given data to do the following:

(a) Construct the group frequency and relative frequency distribution in a table format.

(b) Construct a histogram.

(c) Construct a frequency polygon.

9. The heights (in inches) of the 54 starting players in a men's basketball tournament were as follows:

65	63	77	74	73	67	71	64	74
76	60	66	76	69	63	79	72	61
61	71	66	64	65	72	70	72	76
64	68	68	70	78	71	74	62	70
72	75	76	64	72	70	75	65	68
70	73	67	62	77	68	73	67	66

10. The following data represents the daily high temperatures (in degrees Fahrenheit) for the month of June in North Central America.

68	62	77	76	82	84	69	67	72	74
65	72	80	79	71	74	81	88	90	91
89	82	78	69	69	72	77	81	88	92

11. The following raw data represents the monthly average credit card balance after the 3rd month of receiving the card.

185	75	172	43	76	115	87	160	85	104
123	152	83	58	39	51	59	125	111	70
126	127	78	64	116	113	142	102	93	68
97	142	114	57	122	127	91	85	133	109
115	81	71	59	78	107	105	68	82	162

In Exercises 12-14, determine which type of graph best fits the data. Then construct a table and the correlating graph for the data using Microsoft Excel.

12. The following data are the number of laps completed by the 9th grade swimming team at a local competition.

4	3	7	5	5	6	5	6	6	3	3	9	5	5	7
8	6	3	6	3	4	3	4	5	6	6	7	3	4	3

13. The following data are the scores from last week's math quiz.

7	8	5	4	9	9	6	6	7	9	10	2	3	6	8
4	6	8	3	10	8	7	10	5	9	7	10	5	8	4

14. The following data are the starting salaries of Recent College Graduates.

Salary (thousands of dollars) x	Number of Graduates f
$31-35$	75
$36-40$	140
$41-45$	320
$46-50$	275
$51-55$	130
$56-60$	45
$61-65$	25

6.2 Measures of Central Tendency

A local florist had the following daily sales over a one-week period:

$$\$249, \quad \$310, \quad \$590, \quad \$180, \quad \$268, \quad \$247$$

Since every day was different, the florist would like to have a single number to serve as a kind of representative of the week. A number that would act as an "average" or "middle" number. This is known as a **measure of central tendency**. Four such measures are discussed in this section.

Mean

Finding the average of a set of numbers is the most common measure of central tendency. This common measure is called the **mean**. When computing the mean, you first find the sum of the sample and then divide the sum by the total number of sample items.

$$\text{Mean} = \frac{x_1 + x_2 + x_3 + ... + x_n}{n} = \frac{\Sigma x}{n}$$

where **sigma**, Σ, is the symbol for summation, Σx is the sum of the sample, and n is the number of items.

Example 6.4. From above, a local florist had the following daily sales over a one-week period:

$$\$249, \quad \$310, \quad \$590, \quad \$180, \quad \$268, \quad \$247$$

Find the mean daily sales for this one-week period.

Solution. To find the mean we will use the formula

$$\text{Mean} = \frac{\Sigma x}{n} = \frac{\$1,844}{6} = \$307.44$$

So, the local florist averaged around $\$307.44$ every day. ■

Look back at Example 6.3. See how some data values are identical. Since they are identical, we can use multiplication when computing the mean to simplify the summation of items. When data values occur more than once you can organize the data set into a *frequency distribution*. To calculate the mean using a frequency distribution, the following formula can be used,

$$\text{Mean} = \frac{\Sigma xf}{n},$$

where

- x is the data value,
- f is the frequency of that value,
- Σxf is the sum of each value's product obtained by multiplying each value by its frequency, and
- n is the total frequency of the distribution.

Example 6.5. Find the mean salary for a company that pays annual salaries to its employees shown below.

Salary, x	Number of Employees, f
$\$24,000$	9
$\$28,000$	13
$\$32,000$	16
$\$37,000$	11
$\$49,000$	4
$\$70,000$	2

Solution. We can set up the work as follows:

Salary, x	Number of Employees, f	Salary $\times$ Number
$\$24,000$	9	$\$216,000$
$\$28,000$	13	$\$364,000$
$\$32,000$	16	$\$512,000$
$\$37,000$	11	$\$407,000$
$\$49,000$	4	$\$196,000$
$\$70,000$	2	$\$140,000$
Total	55	$\$1,835,000$

$$\text{Mean salary} = \frac{\$1,835,000}{55} = \$33,363.64$$

■

Median

When determining the middle of the pack, the median is the center. In other words, the **median** is the number dividing the group of numbers into two parts, a lower half and a upper half. In order to find the median, follow these three steps:

Step 1: Arrange the numbers in order, from smallest to largest.

Step 2: If the total is odd, the median is the number in the middle of the list.

Step 3: If the total is even, the median is the mean of the two middle numbers.

Example 6.6. Find the median for each of the following groups of data:

(a) $74,\ 80,\ 88,\ 85,\ 78$

(b) $58,\ 64,\ 1,\ 3,\ 5,\ 15,\ 18,\ 49,\ 24,\ 37$

Solution. (a) Arrange the data values in order, from smallest to largest. The total number of data values is 5, which is odd. Thus, the median is the middle number.

$$74,\ 78,\ 80,\ 85,\ 88$$

The median is 80.

(b) Arrange the data items is order, from the smallest to the largest. The number of data values in the list is 10 which is even. Thus, the median is the mean of the two middle data values.

$$1,\ 3,\ 5,\ 15,\ 18,\ 24,\ 37,\ 49,\ 58,\ 64$$

$$\text{Median} = \frac{18+24}{2} = \frac{42}{2} = 21$$

The median is 21. ■

Mean vs. Median

To find the average, we find the mean. To find a group's middle number, we find the median. But which is better? Let's compare the two measurements.

Example 6.7. Six employees earn annual salaries of

$$\$24,700,\ \ \$25,800,\ \ \$26,500,\ \ \$27,600,\ \ \$28,000,\ \ \$100,000$$

(a) Find the median annual salary for the six people.

(b) Find the mean annual salary for the six people.

Solution. (a) To compute the median, first arrange the salaries in order:

$$\$24,700,\ \ \$25,800,\ \ \$26,500,\ \ \$27,600,\ \ \$28,000,\ \ \$100,000$$

Because the list contains an even number of data values, the median is the mean of the two middle items.

$$\text{Median} = \frac{\$26,500 + \$27,600}{2} = \frac{\$54,100}{2} = \$27,050$$

The median annual salary is $\$27,050$.

(b) We find the mean annual salary by adding the six annual salaries and dividing by 6.

$$\begin{aligned} \text{Mean} &= \frac{\$24,700 + \$25,800 + \$26,500 + \$27,600 + \$28,000 + \$100,000}{6} \\ &= \frac{\$232,600}{2} \\ &= \$38,766.67 \end{aligned}$$

The mean annual salary is around $\$38,767$. Looking at the salaries, we can see that the mean does not give an accurate description of the average salary. Instead, the median allows people to see a fairly unbiased description of the employees' salaries. ■

Mode

The third measure of central tendency is the mode. Different from the mean and median, the **mode** is the value that occurs most often in a data set. Let's say, if 8 students earned scores on a business law examination of

$$\mathbf{81},\ 74,\ 90,\ 67,\ \mathbf{81},\ 90,\ 100,\ \mathbf{81}$$

we notice that more students earned the score of 81 than any other score. Thus, the mode of the list is 81.

Example 6.8. Find the mode for the following group of data:

$$6,\ 3,\ 8,\ 6,\ 1,\ 10,\ 2,\ 5$$

Solution. The number 6 occurs more often than any other. Therefore, 6 is the mode. ■

Midrange

The fourth common measure of central tendency is **midrange**. It is obtained by adding the lowest and highest data values together and dividing the sum by 2.

$$\text{Midrange} = \frac{\text{lowest data value} + \text{highest data value}}{2}$$

Example 6.9. In the Eastern Division of Minor League Baseball during the 2007 season, the New Jersey Stones baseball team had the greatest payroll, a record $\$187,394,100$ (median salary: $\$2,847,219$). The Florida Sea Gulls were the worst paid team, with a payroll of $\$12,378,210$ (median salary: $\$389,000$). Find the midrange for the annual payroll of Eastern division Minor League Baseball teams in 2007.

Solution.

$$\begin{aligned}\text{Midrange} &= \frac{\text{lowest annual payroll} + \text{highest annual payroll}}{2}\\ &= \frac{\$12,378,210 + \$187,394,100}{2}\\ &= \frac{\$199,772,310}{2}\\ &= \$99,886,155\end{aligned}$$

The midrange for the annual payroll of minor league baseball teams in 2007 was $\$99,886,155$. ■

Example 6.10. Let's compare the four measures of central tendency (mean, median, mode, midrange) by computing your six exam grades in a math course. The scores are as follows:

$$52,\ 93,\ 85,\ 93,\ 81,\ \text{and } 86$$

Use $90-100=$ A, $80-89=$ B, $70-79=$ C, $60-69=$D, below $60=$ F.

Solution. (a) The mean is the sum of the data items divided by the number of items.

$$\text{Mean} = \frac{52+93+85+93+81+86}{6} = \frac{490}{6} \approx 81.67$$

Using the mean your final course grade is a B.

(b) Arrange the six data items in order

$$52,\ 81,\ 85,\ 86,\ 93,\ 93$$

Because the number of data items is even, the median is the mean of the two middle items.

$$\text{Median} = \frac{85+86}{2} = \frac{171}{2} = 85.5$$

Using the median, your final course grade is a B.

(c) The mode is the data value that occurs most frequently. Because 93 occurs most often, the mode is 93. Using the mode, your final course grade is an A.

(d) The midrange is the mean of the lowest and highest data values.

$$\text{Midrange} = \frac{52+93}{2} = \frac{145}{2} = 72.5$$

Using the midrange, your final course grade is a C. ■

Exercises

For each list of data in Exercises 1-3, calculate (a) the mean,(b) the median, and (c) the mode. Round mean values to the nearest tenth.

1. $6,\ 10,\ 11,\ 15,\ 23,\ 26,\ 28,\ 34,\ 10$
2. $12,567;\ 14,357;\ 16,457;\ 19,826;\ 14,357;\ 18,274;\ 11,246;\ 21,934$
3. $78,\ 93,\ 78,\ 45,\ 78,\ 26,\ 78,\ 91,\ 78,\ 73,\ 78,\ 56$

In Exercises 4-5, find the mean, median, and mode for the data items in the given frequency distribution.

4.

Score, x	1	2	3	4	5	6	7	8
Frequency, f	2	4	5	5	7	6	4	3

5.

Score, x	1	2	3	4	5	6	7	8	9	10
Frequency, f	1	1	3	5	6	10	9	7	5	4

In Exercises 6-7, calculate the midrange.

6. $7,\ 5,\ 3,\ 2,\ 8,\ 6,\ 1,\ 3$
7. $100,\ 110,\ 140,\ 170,\ 180,\ 120$
8. A set of 10 quiz scores from a math class has a mean of 7.6. When the professor checks the grades at a later date, she finds one score was not recorded. The scores that were recorded are $7,\ 8.5,\ 7,\ 6,\ 5,\ 9,\ 7.5,\ 8,\ 7.75$ What is the missing score?
9. An instructor records 5 exam scores in a psychology class. The mean of those scores is 72.5. The instructor later discovers that a score of 86 was incorrectly recorded as a 68. What should the correct mean be?
10. Consider the data set $4,\ 7,\ 3,\ 6,\ 10,\ 5$.
    (a) Find the mean, median, and midrange of the data set.
    (b) Modify the data by adding 5 to each value. Find the mean, median, and midrange of the new data set.
    (c) How do the means from parts (a) and (b) compare?

(d) How do the medians from parts (a) and (b) compare?

(e) How do the midranges from parts (a) and (b) compare?

11. Consider the data set $4,\ 7,\ 3,\ 6,\ 10,\ 5$.

    (a) Find the mean, median, and midrange of the data set.

    (b) Modify the data by multiplying each value by 3. Find the mean, median, and midrange of the new data set.

    (c) How do the means from parts (a) and (b) compare?

    (d) How do the medians from parts (a) and (b) compare?

    (e) How do the midranges from parts (a) and (b) compare?

6.3 Measures of Dispersion

Typically, Houston, Texas, and Honolulu, Hawaii have an average temperature of $75°$. But even though their average may be similar, their lows and highs are not. In Houston, the evenings may get down into the 40s, while it's highs range in the 100s. Meanwhile, Honolulu's range is between $60°$ and $90°$. Thus, the measure of central tendency does not give the whole story of the data. A **measure of dispersion** is another descriptive measurement used to describe the spread of values in a data set. Two common measures are the *range* and the *standard deviation*.

Range

A quick and straightforward measure of dispersion, the **range** shows the difference between the highest and lowest data values in a data set; thus the spread of the data.

$$\text{Range} = (\text{greatest set value}) - (\text{least set value})$$

Example 6.11. The five countries with the largest labor forces are below.

- China: 834 million
- Taiwan: 679 million
- India: 531 million
- U.S.: 175 million
- Brazil: 86 million

Find the range of workers, in millions, for these five countries.

Solution.

$$\text{Range} = (\text{greatest set value}) - (\text{least set value}) = 834 - 86 = 748 \text{ million}$$

The range is 748 million workers. ■

Example 6.12. Looking back at the temperatures of Houston and Honolulu, Houston's highest temperature is $115°$ while its lowest temperature is $35°$. Honolulu's highest temperature is $93°$ while its lowest temperature is $60°$. Find the range of each and compare.

Solution. Houston: $115° - 35° = 80°$ difference Honolulu: $93° - 60° = 33°$ difference Thus, Honolulu has a smaller variance of temperatures and has a greater chance of hitting $75°$ ■.

Standard Deviation

The second measure of dispersion is found by determining each data value's deviation from the mean. It's called **standard deviation**. To find the standard deviation, follow these six steps:

STEP 1: Calculate the mean of the data set.

STEP 2: Calculate the deviation of each data value from the mean:

$$\text{data value} - \text{mean}$$

STEP 3: Square each deviation:

$$(\text{data value} - \text{mean})^2$$

STEP 4: Sum the squared deviations:

$$\Sigma(\text{data value} - \text{mean})^2$$

STEP 5: Divide the sum in step 4 by $n - 1$, where n represents the number of data values.

$$\frac{\Sigma(\text{data value} - \text{mean})^2}{n-1}$$

STEP 6: Take the square root of the quotient in step 5. This value is the standard deviation for the data set.

$$\text{Standard deviation} = \sqrt{\frac{\Sigma(\text{data value} - \text{mean})^2}{n-1}}$$

Example 6.13. Find the standard deviation from the set of data items

$$800,\ 473,\ 158,\ 119,\ \text{and } 64$$

Solution. STEP 1: Calculate the mean.

$$\text{Mean} = \frac{\Sigma x}{n} = \frac{800 + 473 + 158 + 119 + 64}{5} = \frac{1614}{5} = 322.8$$

STEP 2: Find the deviation from the mean.

Data value	800	473	158	119	64
Deviation	$800 - 322.8 = 477.2$	150.2	-164.8	-203.8	-258.8

STEP 3: Square each deviation.

Data value	Deviation	Square each deviation
800	477.2	227,719.84
473	150.2	22,560.04
158	-164.8	27,159.04
119	-203.8	41,534.44
64	-258.8	66,977.44

STEP 4: Sum the squared deviations.

$$\begin{aligned}\Sigma(\text{data value} - \text{mean})^2 &= 227{,}719.84 + 22{,}560.04 + 27{,}159.04 \\ &\qquad + 41{,}534.44 + 66{,}977.44 \\ &= 382{,}650.8\end{aligned}$$

STEP 5: Divide the sum in step 4 by $n - 1$, where n represents the number of data values.

$$\frac{\Sigma(\text{data value} - \text{mean})^2}{n-1} = \frac{382{,}650.8}{5-1} = 95{,}662.7.$$

STEP 6: Take the square root of the quotient in step 5. This value is the standard deviation for the data set.

$$\text{Standard deviation} = \sqrt{\frac{\Sigma(\text{data value} - \text{mean})^2}{n-1}} = \sqrt{95{,}662.7} = 309.3$$

■

Exercises

1. Find the range for the data items below.

$$1,\ 4,\ 8,\ 17,\ 19,\ 20$$

In Exercises 2-4, a group of data items and their mean are given.

(a) Find the deviation from the mean for each of the data items.

(b) Find the sum of the squared deviations in part (a).

2. $4,\ 6,\ 8,\ 13,\ 19,\ 28$; Mean $= 13$

3. $83,\ 87,\ 89,\ 94,\ 97$; Mean $= 90$

4. $38,\ 35,\ 31,\ 42,\ 47$; Mean $= 38.6$

In Exercises 5-7, find (a) the mean; (b) the deviation from the mean for each data item; and (c) the sum of the squared deviations in part (b).

5. $75,\ 85,\ 80,\ 75,\ 90$
6. $148,\ 155,\ 157,\ 162,\ 163$
7. $3.25,\ 4.50,\ 3.75,\ 4.10,\ 2.90$

In Exercises 8-10, find the standard deviation for each group of data items. Round answers to two decimal places.

8. $15,\ 16,\ 17,\ 18,\ 19$
9. $3,\ 3,\ 3,\ 4,\ 5,\ 5,\ 5$
10. $20,\ 22,\ 23,\ 24,\ 26$
11. Consider the data set $4,\ 7,\ 3,\ 6,\ 10,\ 5$.
    (a) Find the standard deviation of the data set.
    (b) Modify the data by adding 5 to each value. Find the standard deviation of the new data set.
    (c) How do your answers from parts (a) and (b) compare?
12. Consider the data set $4,\ 7,\ 3,\ 6,\ 10,\ 5$.
    (a) Find the standard deviation of the data set.
    (b) Modify the data by multiplying each value by 3. Find the standard deviation of the new data set.
    (c) How do your answers from parts (a) and (b) compare?

6.4 Normal Distribution

Did you know that our heights are on the rise? In one million B.C., the mean height for women was 4 feet 2 inches while the mean height for men was 4 feet 6 inches. Today, however, it is believed that because of improved diets and medical care, the mean height for women rose to 5 feet 5 inches and the men's mean height to 5 feet 10 inches.

Let's suppose that a researcher selects a random sample of 100 adult men, measures their heights, and constructs a histogram. Figure 6.5 depicts the results of the research and illustrate what happens if the sample size increases. As the size increases, you can see that the middle splits the height bars in half. Such a histogram is called **symmetric**. So, as the sample size increases, so does the graph's symmetry. Eventually, the histogram would approach what is called the **normal distribution**, shown in Figure 6.5(d). Its shape, or distribution is also called the **bell curve** or the **Gaussian distribution**, named for the German mathematician

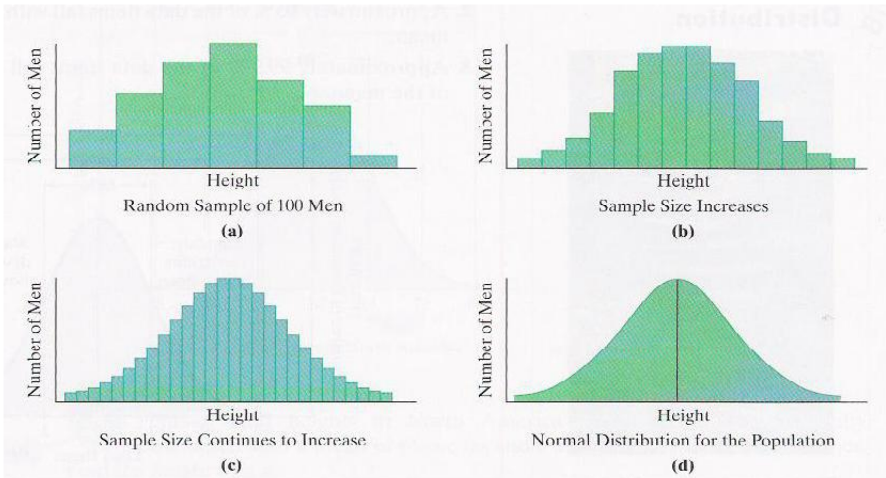


Figure 6.5: Histograms approaching Normal distribution

Carl Friedrich Gauss (1777-1855).

As shown above, the normal distribution is bell shaped and symmetric about its vertical line in the center. In addition to the vertical line, the mean, median, and mode of a normal distribution *are all equal* and also located at the center of the distribution.

Besides the center, the shape of the normal distribution depends on the mean and the standard deviation. Figure 6.6 illustrates three normal distributions with the same mean but different standard deviations. Thus, as the standard deviation increases, the distribution becomes more dispersed, or spread out. However, the distribution still retains its symmetric bell shape.

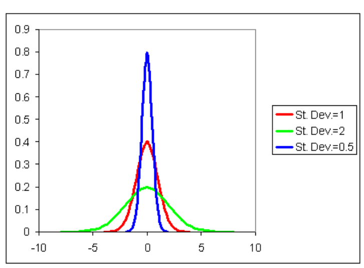


Figure 6.6: Normal Distributions with the same mean but different standard deviations

The Standard Deviation and z-Scores in Normal Distributions

The normal distribution is also divided into the standard deviation and can be summarized by the **Empirical Rule**, also called the $\mathbf{68-95-99.7}$ **Rule**. The rule is as follows in both directions from the mean.

68-95-99.7 Rule

- Approximately 68% of the data values fall within 1 standard deviation of the mean.
- Approximately 95% of the data values fall within 2 standard deviations of the mean.
- Approximately 99.7% of the data items fall within 3 standard deviations of the mean.

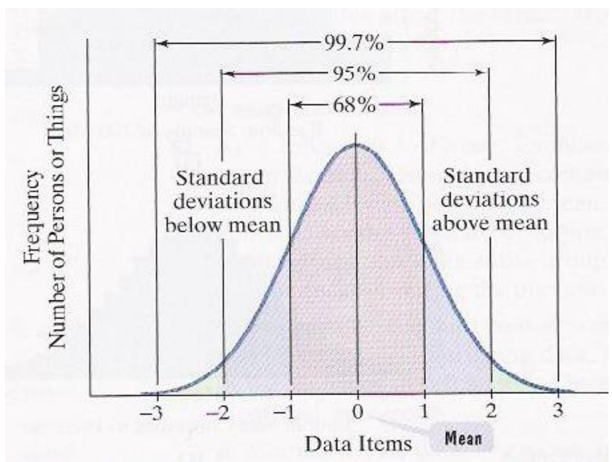


Figure 6.7: Normal Distribution graph with Empirical Rule

From Figure 6.7, you can see that very few values lie more than 3 standard deviations above or below the mean. As the curve moves away from the mean, center, it falls rapidly until it gradually levels out toward the horizontal axis. And although it may look like the curve comes quite close to the axis, its range of the normal distribution is infinite.

Example 6.14. Female adult heights in North America are approximately normally distributed with a mean of 66 inches and a standard deviation of 3 inches. Find the height that is

(a) 2 standard deviations below the mean.

(b) 3 standard deviations above the mean.

Solution. (a) Since the deviation is below the mean, we will subtract the deviation from the mean.

$$\text{Height} = \text{mean} - 2 \times \text{ standard deviation} = 66 - 2 \times 3 = 60$$

A height of 60 inches is 2 standard deviations below the mean.

(b) Since the deviation is above the mean, we will add the deviation to the mean.

$$\text{Height} = \text{mean} + 3 \times \text{standard deviation} = 66 + 3 \times 3 = 75$$

A height of 75 inches is 3 standard deviations above the mean. ■

Example 6.15. Use the distribution of female adult heights in Figure 6.8 to find the percentage of women in North America with heights

(a) Between 63 inches and 69 inches.

(b) Between 66 inches and 69 inches.

(c) Above 72 inches.

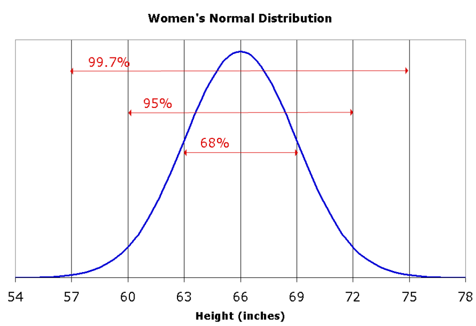


Figure 6.8: Female Adult Heights

Solution. (a) The $68 - 95 - 99.7$ Rule states that approximately 68% of the data values fall within 1 standard deviation, 3, of the mean, 66.

$$\text{mean} - 1 \times \text{standard deviation} = 66 - 1 \times 3 = 66 - 3 = 63$$

$$\text{mean} - 1 \times \text{standard deviation} = 66 + 1 \times 3 = 66 + 3 = 69$$

The Figure 6.8 shows that 68% of female adults have heights between 63 inches and 69 inches.

(b) Because of the distribution's symmetry, the percentage with heights between 63 inches and 66 inches is the same as the percentage with heights between 66 inches and 69 inches. Thus, since Figure 6.8 shows that the percentage from 63 inches to 69 inches is 68%, half would be 34% of female have heights between 66 inches and 69 inches.

(c) We know a height of 72 inches is 2 standard deviations or 6 inches above the mean of 66 inches. By the $68 - 95 - 99.7$ Rule, we also know that approximately 95% of values fall within 2 standard deviations. Since we need the percentage above 72 inches we can subtract 95% from 100%.

$$100\% - 95\% = 5\%$$

However this 5% covers two regions – that greater than 2 standard deviations above the mean and that greater than 2 standard deviations below the mean. Thus 2.5% of females have heights above 72 inches. ■

In a normal distribution, a **z-score** describes how many standard deviations a particular data value lies above or below the mean. The z-score can be obtained by:

$$z\text{-score} = \frac{\text{data value} - \text{mean}}{\text{standard deviation}}$$

If the z-score is positive, the data value is above the mean; negative, below the mean; zero equal to the mean.

Example 6.16. The mean weight of newborn infants is 6.7 pounds and the standard deviation is 0.5 pound. The weights of newborn infants are normally distributed. Find the z-score for a weight of

(a) 8 pounds.

(b) 6.7 pounds.

(c) 5 pounds.

Solution. We compute the z-score for each weight by using the z-score formula. The mean is 6.7 and the standard deviation is 0.5.

(a) The z-score for a weight of 8 pounds, written z_8, is

$$z_8 = \frac{\text{data item} - \text{mean}}{\text{standard deviation}} = \frac{8 - 6.7}{0.5} = \frac{1.3}{0.5} = 2.6.$$

The z-score of the data value greater than the mean is always positive. So an infant who weighs 8 pounds is 2.6 standard deviations above the mean.

(b) The z-score for a weight of 6.7 pounds is

$$z_{6.7} = \frac{\text{data item} - \text{mean}}{\text{standard deviation}} = \frac{6.7 - 6.7}{0.5} = \frac{0}{0.5} = 0.$$

The z-score for a mean is always 0. A 6.7-pound infant is right at the mean, deviating 0 pounds above or below it.

(c) The z-score for a weight of 5 pounds is

$$z_5 = \frac{\text{data item} - \text{mean}}{\text{standard deviation}} = \frac{5 - 6.7}{0.5} = \frac{-1.7}{0.5} = -3.4.$$

The z-score of a data item less than the mean is always negative. So an infant who weighs 5 pounds is 3.4 standard deviations below the mean. ■

Percentiles and Quartiles

As we have seen, the z-score measures a data value's position in a normal distribution. Another measure of the value's position is its **percentile**. One major example of percentiles are standardized tests. For example, if a score is in the 45th percentile, this means that 45% of the scores are less than this score. So, by definition, if $n\%$ of the values in a distribution are less than a particular data value, the value is in the nth *percentile* of the distribution.

To find the percentile of a given data value, you must

1. Convert the value to z-score.
2. Convert the z-score to a percentile by using the z-score and percentile table (Table 6.3).

Remember that the percentile given is the percentage of data values that are less than the data value in question.

Example 6.17. Let's assume cholesterol levels are normally distributed. For men between 18 and 24 years, the mean is 180 and the standard deviation is 38.5. What percentage of men in this age range have a cholesterol level

(a) less than 230.05?

(b) greater than 206.95?

(c) between 158.83 and 201.18?

Solution. (a) First we need to convert the cholesterol level to a z-score.

$$z_{230.05} = \frac{\text{data item} - \text{mean}}{\text{standard deviation}} = \frac{230.05 - 180}{38.5} = \frac{50.05}{38.5} = 1.3$$

Next, we find the matching percentile to a z-score of 1.3, which is 90.32. This means that 90.32% of men between 18 and 24 have a cholesterol level less than 230.05.

(b) Again, we need to convert the cholesterol level to a z-score.

$$z_{206.95} = \frac{\text{data item} - \text{mean}}{\text{standard deviation}} = \frac{206.95 - 180}{38.5} = \frac{26.95}{38.5} = 0.7$$

Table 6.3: z-Scores and Percentiles

z-score	Percentile	z-Score	Percentile	z-Score	Percentile	z-Score	Percentile
-4.0	0.003	-1.0	15.87	0.0	50.00	1.1	86.43
-3.5	0.02	-0.95	17.11	0.05	51.99	1.2	88.49
-3.0	0.13	-0.90	18.41	0.10	53.98	1.3	90.32
-2.9	0.19	-0.85	19.77	0.15	55.96	1.4	91.92
-2.8	0.26	-0.80	21.19	0.20	57.93	1.5	93.32
-2.7	0.35	-0.75	22.66	0.25	59.87	1.6	94.52
-2.6	0.47	-0.70	24.20	0.30	61.79	1.7	95.54
-2.5	0.62	-0.65	25.78	0.35	63.68	1.8	96.41
-2.4	0.82	-0.60	27.43	0.40	65.54	1.9	97.13
-2.3	1.07	-0.55	29.12	0.45	67.36	2.0	97.72
-2.2	1.39	-0.50	30.85	0.50	69.15	2.1	98.21
-2.1	1.79	-0.45	32.64	0.55	70.88	2.2	98.61
-2.0	2.28	-0.40	34.46	0.60	72.57	2.3	98.93
-1.9	2.87	-0.35	36.32	0.65	74.22	2.4	99.18
-1.8	3.59	-0.30	38.21	0.70	75.80	2.5	99.38
-1.7	4.46	-0.25	40.13	0.75	77.34	2.6	99.53
-1.6	5.48	-0.20	42.07	0.80	78.81	2.7	99.65
-1.5	6.68	-0.15	44.04	0.85	80.23	2.8	99.74
-1.4	8.08	-0.10	46.02	0.90	81.59	2.9	99.81
-1.3	9.68	-0.05	48.01	0.95	82.89	3.0	99.87
-1.2	11.51	0.0	50.00	1.0	84.13	3.5	99.98
-1.1	13.57					4.0	99.997

Now, with a z-score of 0.7, we can find that the matching percentile is 75.80. We know that 75.8% of men have a cholesterol level less than 206.95, but we want the level greater than 206.95. So we subtract 75.8% from 100%

$$100 - 75.8 = 24.2\%$$

So 24.2% of all men have a cholesterol level greater than 206.95.

(c) (i) First we convert each given data item to a z-score.

$$z_{158.83} = \frac{\text{data item} - \text{mean}}{\text{standard deviation}} = \frac{158.83 - 180}{38.5} = \frac{-21.17}{38.5} = -0.55$$

$$z_{201.18} = \frac{\text{data item} - \text{mean}}{\text{standard deviation}} = \frac{201.18 - 180}{38.5} = \frac{21.18}{38.5} = 0.55$$

(ii) Next we find the matching percentile, -0.55 is 29.12% of men have a cholesterol level less than 158.83 and 0.55 is 70.88% of men have a cholesterol level less than 210.18.

(iii) Finally we subtract the lesser percentile from the greater percentile.

$$70.88\% - 29.12\% = 41.76\%$$

Therefore, 41.76% of men have a cholesterol level between 158.83 and 210.18. ■

In addition to percentiles, the **quartile** divides the data values into four equal sets.

- **First quartile** (25th percentile): 25% of the data falls below the first quartile.
- **Second quartile** (50th percentile): 50% of the data falls below the second quartile; equivalent to the median.
- **Third quartile** (75th percentile): 75% of the data falls below the third quartile.

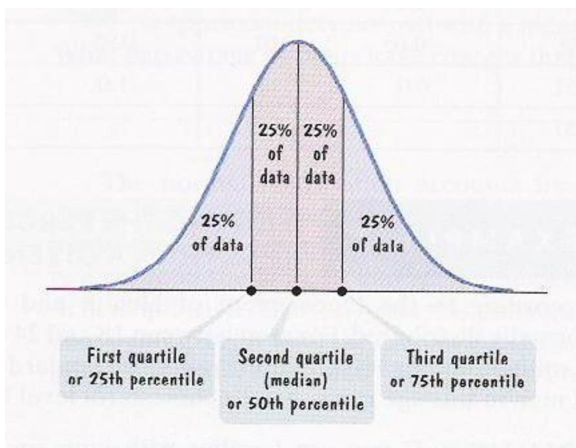


Figure 6.9: Quartile graph

Example 6.18. Find the three quartiles for the 40 given numbers below:

26	31	32	35	36	36	38	39	39	39
41	41	42	42	43	43	44	44	44	45
46	46	46	47	47	47	48	48	49	49
50	50	51	51	52	55	59	59	63	68

Solution. Let's start with Q_2 or the median. Since the total number of items is even we will take the two middle data values, 45 and 46, and divide them by two to find Q_2.

$$Q_2 = \frac{45 + 46}{2} = 45.5$$

There are 20 items (an even number) that are all below Q_2. So we find the two middle items in this set, which are 39 and 41.

$$Q_1 = \frac{39+41}{2} = 40$$

Finally, there are also 20 items that are all above Q_2. So we find the two middle items in this set, which are 49 and 50.

$$Q_3 = \frac{49+50}{2} = 49.5$$

■

The Box Plot

Like many of the graphs from the first section, **the box plot**, also called the **box-and-whisker plot**, displays the relationship between the median, the range, and the first and third quartiles. For a given set of data, the box plot consists of a rectangular box positioned above a numerical scale ranging from Q_1 to Q_3. Q_2, or the median, is indicated within the box. The box also has extending "whiskers", or line segments, that extend to the minimum and maximum data values to the right and left of the box.

Example 6.19. Construct a box plot for the weekly study times data below.

2	4	5	6	7	8	10	12	13	14
14	15	16	16	19	19	19	20	21	21
22	23	26	26	28	29	30	31	34	34
35	35	37	42	45	45	48	50	52	62

Solution. First determine the median:

$$Q_2 = \frac{21+22}{2} = 21.5$$

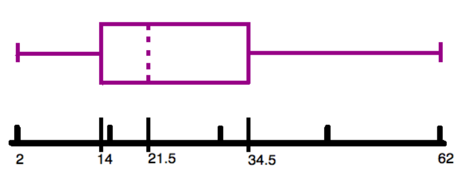
Next find Q_1 and Q_3.

$$Q_1 = \frac{14+14}{2} = 14$$

and

$$Q_3 = \frac{34+35}{2} = 34.5$$

The minimum and maximum values are 2 and 62. The box plot is shown below. ■



Notice the box plot clearly conveys the following:

- central tendency (the location of the median);
- the location of the middle half of the data (the extent of the box);
- dispersion (the range is the extent of the whiskers);
- skewness (the non-symmetry of both the box and the whiskers).

Exercises

Intelligence quotients (IQs) on the Stanford-Binet intelligence test are normally distributed with a mean of 100 *and a standard deviation of* 16. *In Exercises 1-5, use the* $68 - 95 - 99.7$ *Rule to find the percentage of people with IQs*

1. between 84 and 116.
2. above 116.
3. below 100.
4. below 68.
5. above 148.

A set of data items is normally distributed with a mean of 75 *and a standard deviation of* 6. *In Exercises 6-8, convert each data item to a* z*-score.*

6. 63
7. 93
8. 72

A set of data items is normally distributed with a mean of 640 *and a standard deviation of* 80. *In Exercises 9-12, find the data item in this distribution that corresponds to the given* z*-score.*

9. $z = -2.5$
10. $z = 1.5$
11. $z = -1.1$
12. $z = 2.7$

In Exercises 13-16, find the percentage of data items in a normal distribution that lie (a) below and (b) above the given z*-score.*

13. $z = 1.3$
14. $z = -0.8$
15. $z = 0.55$
16. $z = -1.4$

In Exercises 17-20, find the percentage of data items in a normal distribution that lie between

17. $z = 0.3$ and $z = 1.5$.
18. $z = -2.4$ and $z = 0.7$.
19. $z = -0.55$ and $z = 1.2$.
20. $z = -2.5$ and $z = -0.1$

6.5 Scatter Plots, Correlation, and Regression Lines

In the previous sections the data sets only involved single variables. In this section however, we will look at two variables and determine whether or not there is a relationship between them and if so, the strength of that relationship.

Scatter Plots and Correlation

Is there a relationship between education and prejudice? In other words, as a person's education increases does his/her level of prejudice decrease? First notice that the two variables we are using are education and prejudice. The scores below show the two quantities for a random sample of ten people.

Respondent	A	B	C	D	E	F	G	H	I	J
Years of Education (x)	11	4	13	12	8	9	15	10	11	3
Score on prejudice test (y)	0	6	1	2	4	3	0	1	2	9

Besides a table, each data value can be visually displayed using a *scatter plot*. A **scatter plot** is a collection of ordered pairs, or data points. The scatter plot below illustrates the data points by drawing a horizontal axis to represent years of education and the vertical axis to represent scores on a test measuring prejudice. So, for example, respondent B is located to represent 4 years of education on the horizontal axis and 6 on the prejudice test on the vertical axis.

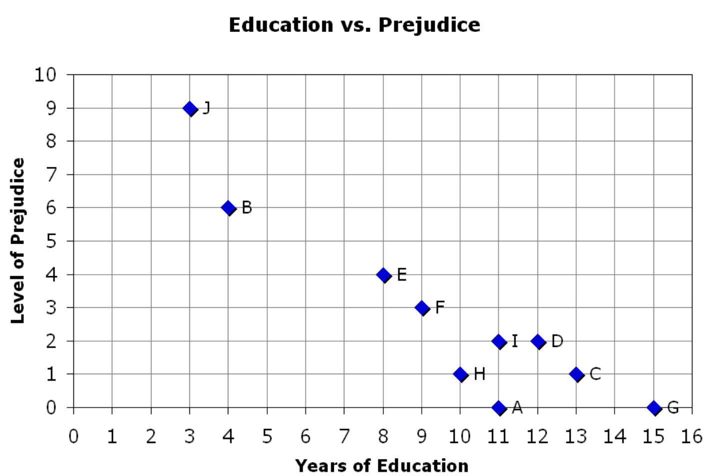


Figure 6.10: A scatter plot

A scatter plot like the one in Figure 6.10 can be used to determine whether two quantities are related. The quantities are **correlated** if there is a clear relationship. In other words, a **correlation** is used to determine if there is a relationship between two variables and, if so, the strength and direction of that relationship.

Regression Lines and Correlation Coefficients

Figure 6.11 shows the education and prejudice scatter plot, with one addition. The straight line that seems to approximately "fit" the data points is called a **regression**

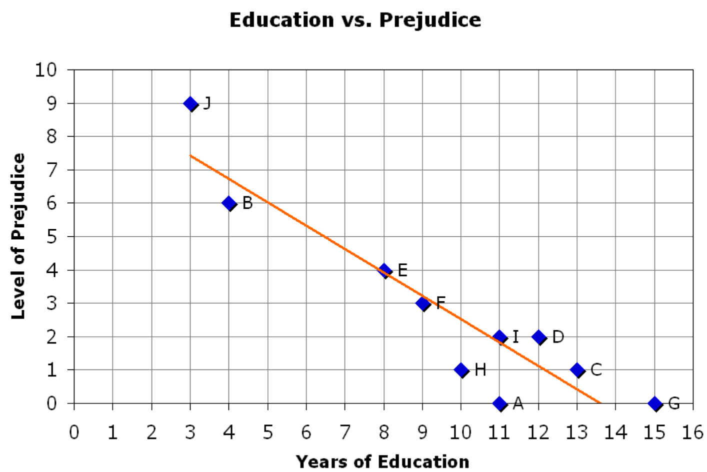


Figure 6.11: A regression line

line. Also known as a "best fit line," the regression line allows the spread of data points to be as close to the line as possible.

The **correlation coefficient**, r, is a measure that describes the strength and direction of the relationship between the variables that lie on or near the regression line. A scatter plot is **positively correlated** if it's data points tend to increase or decrease together. By contrast, scatter plots that are **negatively correlated** contain data points that tend to decrease while the other increases. Besides positive and negative directions, the correlation is also given a strength. For instance, a **perfect positive correlation** indicates that all points in the scatter plot lie precisely on the regression line that rises from left to right. A **perfect negative correlation** shows the points on the regression line, but instead fall from the left to the right. Other strengths include weak and moderate. One last possibility is no correlation where the data points are neither positive, negative, perfect, or weak. See Figure 6.12 for examples.

We can find the correlation coefficient and the equation of the regression line by using a graphing calculator and the following formulas. First we will start with the correlation coefficient, r.

$$r = \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{\sqrt{n(\Sigma x^2) - (\Sigma x)^2}\sqrt{n(\Sigma y^2) - (\Sigma y)^2}}$$

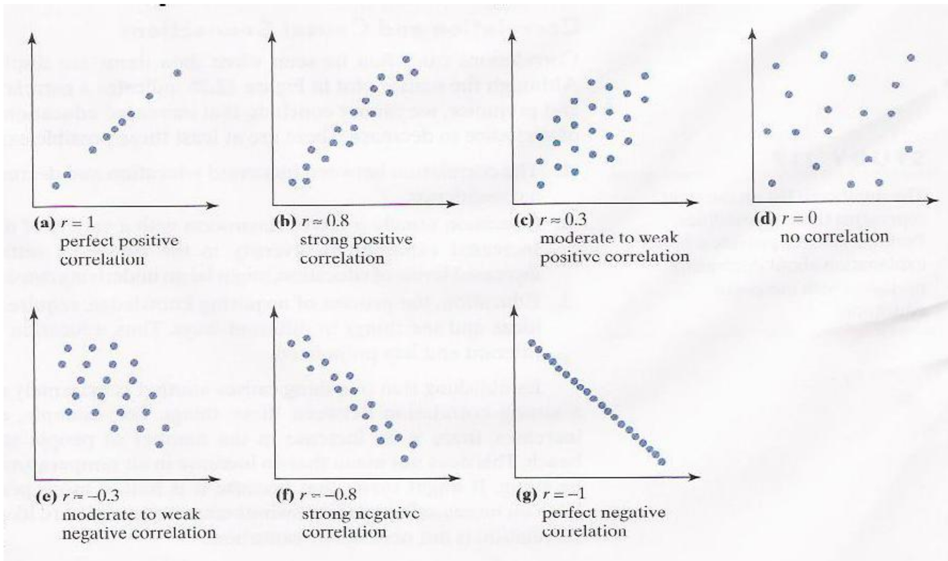


Figure 6.12: Regression and Correlation graphs

where

$$
\begin{aligned}
n &= \text{the number of data points, } (x, y) \\
\Sigma x &= \text{the sum of the } x\text{-values} \\
\Sigma y &= \text{the sum of the } y\text{-values} \\
\Sigma xy &= \text{the sum of the product of } x \text{ and } y \text{ in each pair} \\
\Sigma x^2 &= \text{the sum of the squares of the } x\text{-values} \\
\Sigma y^2 &= \text{the sum of the squares of the } y\text{-values} \\
(\Sigma x)^2 &= \text{the square of the sum of the } x\text{-values} \\
(\Sigma y)^2 &= \text{the square of the sum of the } y\text{-values}
\end{aligned}
$$

Example 6.20. Let's find the correlation coefficient for the education versus prejudice data. Recall the table below.

Respondent	A	B	C	D	E	F	G	H	I	J
Years of Education (x)	11	4	13	12	8	9	15	10	11	3
Score on prejudice test (y)	0	6	1	2	4	3	0	1	2	9

Solution. Organize the work into five columns and add up the totals at the bottom.

x	y	xy	x^2	y^2
11	0	0	121	0
4	6	24	16	36
13	1	13	169	1
12	2	24	144	4
8	4	32	64	16
9	3	27	81	9
15	0	0	225	0
10	1	10	100	1
11	2	22	121	4
3	9	27	9	81
$\Sigma x = 96$	$\Sigma y = 28$	$\Sigma xy = 179$	$\Sigma x^2 = 1050$	$\Sigma y^2 = 152$

We can use these five sums to calculate the correlation coefficient. However, there are still a few computations to be found. The number of data points is 10, so $n = 10$. We also need to find the square of the sum of the x-values and the y-values.

$$(\Sigma x)^2 = (96)^2 = 9216 \text{ and } (\Sigma y)^2 = (28)^2 = 784$$

We are now ready to determine the correlation coefficient, r.

$$\begin{aligned} r &= \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{\sqrt{n(\Sigma x^2) - (\Sigma x)^2}\sqrt{n(\Sigma y^2) - (\Sigma y)^2}} \\ &= \frac{10(179) - (96)(28)}{\sqrt{10(1,050) - (9,216)}\sqrt{10(152) - (784)}} \\ &= \frac{-898}{\sqrt{1,284}\sqrt{736}} \\ &\approx -0.92 \end{aligned}$$

The value for r, approximately -0.92 is close to -1 and indicates a strong negative correlation. This means that the more education a person receives, the less prejudice that person has. ■

Once you have found a correlation coefficient, the next step is to write an equation for the regression line. The formula to find the equation for the regression line is as follows:

$$y = mx + b$$

where

$$m = \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{n(\Sigma x^2) - (\Sigma x)^2} \text{ and } b = \frac{\Sigma y - m(\Sigma x)}{n}$$

Example 6.21. Let's continue with the previous problem and find its regression line equation. Remember that, $\Sigma x = 96$, $\Sigma y = 28$, $\Sigma xy = 179$, $\Sigma x^2 = 1050$, $(\Sigma x)^2 = 9216$.

Solution. Let us first find m and b.

$$m = \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{n(\Sigma x^2) - (\Sigma x)^2} = \frac{10(179) - (96)(28)}{10(1050) - (9216)} = \frac{-898}{1284} \approx -0.70$$

The negative slope agrees with the negative correlation found above. This line falls from left to right, indicating a negative correlation. Now, we find the y-intercept, b.

$$b = \frac{\Sigma y - m(\Sigma x)}{n} = \frac{28 - (-0.70)(96)}{10} = \frac{95.2}{10} \approx 9.52$$

Using $m \approx -0.7$ and $b \approx 9.52$, the equation of the regression line, $y = mx + b$, is

$$y = -0.7x + 9.52$$

where x represents the number of years of education and y represents the score on the prejudice test. ■

Exercises

In Exercises 1-4, make a scatter plot for the given data. Use the scatter plot to describe whether or not the variables appear to be related. Also describe the plots correlation (strong, weak, positive, negative).

1.

Respondent	A	B	C	D	E	F	G
Parent's years of education (x)	11	7	5	10	10	8	9
Child's years of education (y)	11	9	5	14	15	6	15

2.

Respondent	A	B	C	D	E
IQ (x)	100	105	110	115	125
Annual income (y)	28,000	30,000	34,000	38,000	42,000

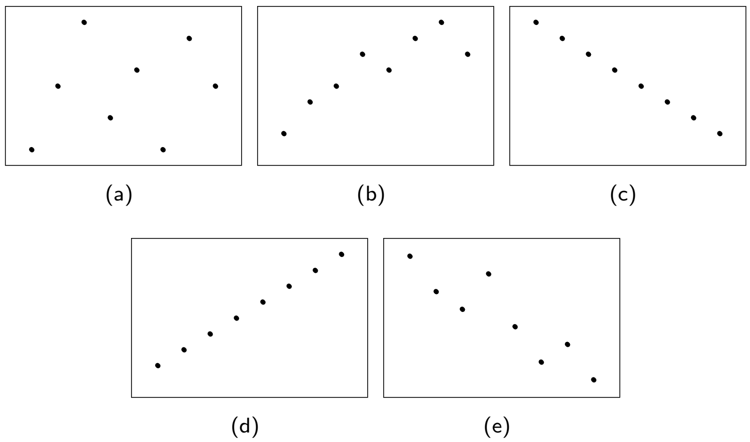
3. The data shows the number of registered automatic hand-guns, in thousands, and the murder rate, in murders per 100,000 for seven randomly selected states.

Automatic hand-gun (x)	12.6	9.3	7.9	4.6	3.6	3.5	3.4
Murder rate (y)	14.1	11.6	12.5	11.1	6.3	7.6	4.6

4. The data shows the number of employed and unemployed female workers, 25 years and older, in thousands, for six selected years in the United States.

Year	2003	2004	2005	2006	2007	2008
Employed (x)	60,085	60,897	62,524	63,134	63,761	64,580
Unemployed (y)	3,029	2,947	2,626	2,380	2,233	2,150

Use the scatter plots shown, labeled (a)-(e), to solve Exercises 5-8.



5. Which scatter plot indicated a perfect positive correlation?
6. Which scatter plot indicated a perfect negative correlation?
7. Which scatter plot shows $r = 0.01$?
8. Which scatter plot shows $r = 0.9$?

Compute the correlation coefficient and the equation for the regression line for the data in

9. Exercise 1.
10. Exercise 2.
11. Exercise 3.

7

Voting Theory

Voting is the cornerstone of a democracy. The United States was founded on the belief that every person's vote counts; therefore, once you turn the age of 18, it is your obligation and sacred right as a citizen of this free country to cast your vote in all elections. We, as voting citizens, must be aware of why it's important that we vote and how our vote really counts. In this chapter we will be addressing different voting systems and the flaws that exist with each system. Doing this will allow us to answer the questions: Why must I vote? Does my vote really count? How does my vote count?

7.1 Voting Systems

In this section we will discuss four of the most popular voting systems and why each is beneficial. In the next section we will look at the flaws of each of these voting systems. The following processes will be discussed:

- Plurality method
- Borda count method
- Plurality with elimination method
- Pairwise comparison method

Plurality Method

In the case that two candidates are running in an election the candidate who receives the most votes wins the election. However, if there are more than two candidates running in the election, one candidate does not necessarily win more than 50% of the votes. We say a candidate has a **majority** of the votes if the candidate wins more than 50% of the votes. If the *plurality method* is used, voters get to vote for one candidate and the candidate who received the greatest number of votes, called a **plurality**, is the winner. This candidate is then called the **plurality candidate**.

The plurality method is beneficial since it only requires the voter to choose their favorite candidate, no ranking of the candidates is involved, and determining the winner is as simple as counting the votes.

Example 7.1. The math club has decided to use the plurality method to select their president. The votes are in and the results are the following:

Alicia – $2,685$ votes

Thomas – $2,348$ votes

Nick – $2,132$ votes

Rachel – $2,456$ votes

Who is elected as president?

Solution. By the plurality method, the candidate with the most (first-place) votes wins. Therefore, Alicia will be elected as math club president. ■

Example 7.2. Three candidates are running for mayor of Nashville: Rogers, Jacobs, and Donaldson. Rogers received $105,670$ votes. Jacobs received $117,561$ votes. Donaldson received $14,787$ votes.

(a) What percentage of the votes did each candidate receive?

(b) Did any candidate receive a majority of the votes?

(c) Under the plurality method, who will be elected?

Solution. (a) A total of

$$105,670 + 117,561 + 14,787 = 238,018$$

votes were collected in the election.

(b) Rogers received $\dfrac{105,670}{238,018} = 44.4\%$ of the vote

Jacobs received $\dfrac{117,561}{238,018} = 49.4\%$ of the vote

Donaldson received $\dfrac{14,787}{238,018} = 6.2\%$ of the vote

No candidate received a majority of the votes because no one received more than 50% of the votes.

(c) Under the plurality method, Jacobs would be elected as mayor since he received the most votes. ■

Borda Count Method

The biggest advantage of the plurality method is that voters are only required to pick their favorite candidate. Other methods, like the *Borda count method*, require voters to rank their candidates in order of preference. The Borda count method assigns points to each candidate. The voter's last choice receives one point, second-to-the-last choice receives two points, and so on, until the voter's first choice receives p points (where there are p candidates total). The votes are collected, the points are totaled, and the candidate who receives the greatest number of points is the winner.

The key advantage of the Borda count method is that it collects more information from the voters than does the plurality method. Rather than voters only choosing their number one candidate, they are also permitted to rank the other candidates and, therefore, providing more information for the tally of the votes.

Example 7.3. Suppose that, again, four people are running for math club president: Alicia, Thomas, Nick, and Rachel. Voters are instructed to rank the four candidates first through fourth, which provides the following preference table. Under the Borda count method, who is selected as math club president?

Candidates	1st place votes	2nd place votes	3rd place votes	4th place votes
Alicia	$2,685$	$3,657$	$2,632$	$1,229$
Thomas	$2,348$	$1,265$	$3,421$	$3,169$
Nick	$2,132$	$2,224$	$3,167$	$2,680$
Rachel	$2,456$	$2,354$	$3,598$	$1,795$

Solution. We need to convert the votes to points by multiplying all first place votes by four, all second place votes by three, all third place votes by two, and all fourth place votes by one. We then add up the points received by each candidate to determine the winner.

Alicia: $(2,685\times 4)+(3,657\times 3)+(2,632\times 2)+(1,229\times 1) = 28,204$ points

Thomas: $(2,348\times 4)+(1,265\times 3)+(3,421\times 2)+(3,169\times 1) = 23,198$ points

Nick: $(2,132\times 4)+(2,224\times 3)+(3,167\times 2)+(2,680\times 1) = 24,214$ points

Rachel: $(2,465\times 4)+(2,354\times 3)+(3,598\times 2)+(1,795\times 1) = 25,913$ points

Using the Borda count method, Alicia is elected math club president since she received the greatest amount of points. ■

Plurality with Elimination Method

Another system for determining a winner involves reducing the number of candidates in the candidate pool until one candidate receives more than 50% of the votes. This is referred to as the *plurality with elimination method* and may require many rounds of voting. The first round of voting consists of the following:

- Each voter votes for their favorite candidate.
- If one candidate wins the majority, that candidate is declared the winner.
- If no candidate receives the majority of votes, the candidate who received the fewest number of votes is eliminated. In the case that there are ties between candidates, all candidates with the fewest number of votes are eliminated.

Another round of voting takes place and if one candidate receives the majority they are declared the winner, otherwise elimination of the candidate with the fewest number of votes continues until one candidate receives the majority and can be declared winner. The plurality with elimination method of voting is a widely used voting system. For example, this method is used to determine the site of the Olympic Games every two years.

Sometimes different rules are used to determine which candidates continue on to the second or later rounds of voting. For example, one variation states after the first round of voting, the election may only permit the top two candidates to continue on to the second round. In the case that only two candidates are remaining for the second round of voting, the second round is called a **runoff election**. However, one common problem with this variation may occur when the two runoff candidates do not receive a composite majority of votes. For example, let's say that of 20 candidates, none receive a majority of votes during the first round of an election so a runoff election is held. The top two candidates are Johnson, who received 17.2% of the votes, and Reed, who received 16.8% of the votes. Notice that the top two candidates only won 34.0% of the votes which is just barely over one-third of the votes. Nearly two-thirds of the voters did not vote for either of the top two candidates.

In our examples, we will display the voters' rankings in a *preference table*. The first round consists of voters casting their votes for their favorite candidate. If no majority is reached, the candidate with the least number of votes is eliminated. The first-place votes of the eliminated candidate go to the voters' second-place choice. This replicates the production of another ballot without having to hold another voting round.

Example 7.4. Anna(A), Brandon (B), Charles (C), and Danielle (D) are running for department president. Fifteen voters are asked to rank the candidates in order of their personal preference. The votes received are summarized in the following table with each column representing one voter's ballot. Using the plurality with elimination method, who is elected as department president?

Ranking	Ballots																
1st	A	B	B	D	A	C	D	B	C	D	A	D	B	B	C	B	A
2nd	C	C	C	B	C	A	A	D	B	A	C	B	D	A	A	A	C
3rd	B	A	A	C	B	A	C	A	D	C	B	C	A	D	D	D	D
4th	D	D	D	A	D	B	B	C	A	B	D	A	C	C	B	C	B

Solution. Notice that some voters' ballots are identical so we are able to condense the preference table by combining the columns of those with identical rankings. The following table shows this condensed version of the results. The number at the top of each column is the number of ballots receiving this ranking.

	Number of Ballots								
Ranking	3	2	2	2	2	2	1	2	1
1st	A	B	D	C	D	B	C	B	A
2nd	C	C	B	A	A	D	B	A	C
3rd	B	A	C	D	C	A	D	D	D
4th	D	D	A	B	B	C	A	C	B

In order to use the plurality with elimination method, we must first determine how many first place votes were received by each candidate. For example, to count up Anna's first-place votes, we count up the numbers at the top of the columns with an A in the row labeled '1st'.

	Anna	Brandon	Charles	Danielle
1st place votes	4	6	3	4

Since no candidate received a majority of the votes, Charles, who has the fewest first-place votes, is eliminated from the election.

Now, a new preference table of only Anna, Brandon, and Danielle must be constructed. The following table illustrates what happens to the voters' ballots who did not place Charles in the 4th rank. Using the first column of the above preference table as an example, the voters' ballots of the first column will be changed to,

1st	A
2nd	B
3rd	D

Charles was the second choice of these voters, but since Charles has been eliminated from the election, their third choice becomes their second choice and their fourth choice becomes their third choice. The new preference table, after making the changes to account for Charles' elimination is shown in the following table:

	Number of Ballots								
Ranking	3	2	2	2	2	2	1	2	1
1st	A	B	D	A	D	B	B	B	A
2nd	B	A	B	D	A	D	D	A	D
3rd	D	D	A	B	B	A	A	D	B

Again, we need to count the number of first-place votes received by each candidate to see if someone received a majority. Here are the results:

	Anna	Brandon	Danielle
1st place votes	6	7	4

Once again, no candidate received a majority so the candidate with the least number of first-place votes, Danielle, must be eliminated. The choices of those who had ranked Danielle first or second must again be retabulated. The following preference table shows these results.

Ranking	Number of Ballots								
	3	2	2	2	2	2	1	2	1
1st	A	B	B	A	A	B	B	B	A
2nd	B	A	A	B	B	A	A	A	B

Now we count the number of first-place votes received by each candidate to determine the winner.

	Anna	Brandon
1st place votes	8	9

With nine votes, Brandon is elected as department president. ■

Pairwise Comparison Method

The **pairwise comparison method**, also called the **Condorcet method**, uses rankings to compare pairs of candidates. The voters are required to make a preferred choice between every possible pair of candidates. For example, if the candidates are Riggs, Jones, and Garcia, voters must vote three times to make a decision between Riggs versus Jones, Riggs versus Garcia, and Jones versus Garcia. Rather than completing this process using three separate ballots, voters are only required to rank their candidates in order of preference. For example, one voter ranks the candidates as,

Jones
Riggs
Garcia

This ranking would allow us to assume that the voter would choose Jones over Riggs in the Riggs-versus-Jones case, Jones over Garcia in the Jones-versus-Garcia case, and Riggs over Garcia in the Riggs-versus-Garcia case.

Using the rankings of the voters, we go through every possible pairing to determine which candidates are preferred. Points are assigned based on how well each candidates does with respect to the others. If candidate X is preferred to another candidate Y by most voters, candidate X gets one point. If the candidates tie, each candidate receives $\frac{1}{2}$ point. Eventually, the candidate who receives the greatest amount of points is declared the winner.

Example 7.5. Charles decides to withdraw himself from the department president election leaving only three candidates: Anna, Brandon, and Danielle. The 17 voters are again asked to rank the candidates first through third. When Charles withdraws himself, the preference tables are altered in the same way they were in the previous example when Charles was eliminated in the first round. Under the pairwise comparison method, who wins the election?

Ranking	Number of Ballots					
	3	4	2	3	2	3
1st	A	B	D	A	D	B
2nd	B	A	B	D	A	D
3rd	D	D	A	B	B	A

Solution. We must consider three pairs of candidates:

Anna vs. Brandon

Brandon vs. Danielle

Anna vs. Danielle

To make these comparisons, we only consider the part of the preference table that includes the two candidates being compared.

First, we will compare Anna and Brandon by deleting Danielle from the preference table:

Ranking	Number of Ballots					
	3	4	2	3	2	3
1st	A	B	B	A	A	B
2nd	B	A	A	B	B	A

As we can see in the above table, Anna received 8 first-place votes and Brandon received 9 first-place votes. So, we say that Brandon is preferred to Anna by a margin of 9 to 8. Therefore, Brandon receives 1 point.

Next, we compare Anna to Danielle by removing Brandon from the preference table:

Ranking	Number of Ballots					
	3	4	2	3	2	3
1st	A	A	D	A	D	D
2nd	D	D	A	D	A	A

Anna received 10 first-place votes and Danielle received 7 first-place votes so Anna is preferred to Danielle by a margin of 10 to 7 and, therefore, receives 1 point.

Lastly, we will compare Brandon to Danielle by removing Anna from the preference table:

Ranking	Number of Ballots					
	3	4	2	3	2	3
1st	B	B	D	D	D	B
2nd	D	D	B	B	B	D

Brandon received 10 first-place votes and Danielle received 7 first-place votes. So, Brandon is preferred to Danielle by a margin of 10 to 7 so Brandon receives 1 point.

Since this was our last comparison, the final point tally is as follows:

	Anna	Brandon	Danielle
points	1	2	0

By the pairwise comparison method, Brandon becomes the department chair because he received the most points. The results of the pairwise comparisons are summarized in the preference table below. The candidates' names are listed across the top of the table and along the left-hand side. Reading across a candidate's row will depict how the candidate compared with the others. The candidates' vote totals appear in the parentheses and the points awarded are in boldface.

Candidate	Anna	Brandon	Danielle	Point Total
Anna	N/A	$\mathbf{0}(8-9)$	$\mathbf{1}(10-7)$	**1**
Brandon	$\mathbf{1}(9-8)$	N/A	$\mathbf{1}(10-7)$	**2**
Danielle	$\mathbf{0}(7-10)$	$\mathbf{0}(7-10)$	N/A	**0**

■

A summarized preference table can also be used to sum up the results of an election in which the Borda count method was used.

Breaking a Tie

When using the same preference table, it is possible for the four voting systems we have discussed to produce different winners. We will discuss this possibility further in the next section. It is also possible for any of the four methods to produce a tie if the preferences of the voters are perfectly balanced. In this case, a tie can be broken by making a random choice (for instance by flipping a coin) or by bringing in another voter.

Other methods may also be used to break a tie depending on which voting system is being used. For example, when using the Borda count method, a tie could be broken by choosing the candidate who has received the most first-place votes. However, the different methods of breaking a tie could produce different winners. Thus it is important that the tie-breaking method is determined before the election begins.

Exercises

1. Four candidates ran for vice-president of the PTA. The following preference table lists the candidates and the votes received by each:

Candidate	Votes
Holen	32
Rodney	28
James	13
Peterson	48
Total	121

(a) Under the plurality method, who is the winner of the election?

(b) What percentage of the votes did each candidate receive?

(c) If the PTA required that the candidate win by a majority, how many votes would be required that the winning candidate receive?

(d) Did any candidate receive a majority? If yes, who?

2. Six candidates are running for governor of Nebraska. The following preference table lists the candidates and the votes received by each:

Candidate	Votes
Burndt	$89,371$
Collins	$45,670$
Humphrey	$87,345$
Richardson	$108,487$
Salgren	$103,211$
Thompson	$56,934$
Total	$491,018$

(a) Under the plurality method, who is the winner of the election?

(b) What percentage of the votes did each candidate receive?

(c) If the state required that the candidate win by a majority, how many votes would be required that the winning candidate receive?

(d) Did any candidate receive a majority? If yes, who?

3. A committee of 14 students are voting on the location for their end-of-year party. Their three choices are the Holiday Inn (H), the park (P), or the bowling alley (B). Their preferences appear in the following preference table:

	Number of members			
Ranking	5	4	3	2
1st	H	B	P	B
2nd	P	H	B	P
3rd	B	P	H	H

(a) If the committee required that the winning location must receive a majority of the *first-place* votes, how many votes would be required to win?

(b) Calculate the number of first-place votes received by each candidate.

(c) Using the plurality method, determine which location will be selected.

4. The head of the advertising department wants to add another form of advertisement. His committee of 37 persons will be voting by ranking the following options in their order of preference: magazine (M), billboard (B), or commercial (C). Their preferences are summarized in the following table:

Ranking	Number of members					
	9	3	7	6	2	10
1st	M	B	C	B	M	C
2nd	C	M	B	C	B	M
3rd	B	C	M	M	C	B

(a) If the head of the department requires that the winning form of advertisement must receive a majority of the *first-place* votes, how many votes would be required to win?

(b) Calculate the number of first-place votes received by each advertisement form.

(c) Using the plurality method, determine which form of advertisement will be selected.

5. The seminar planning committee has decided to use the Borda count voting method to choose their next seminar location from the following options: Orlando (O), Nashville (N), or San Francisco (S). The preferences of the 9 members of the committee appear in the following table.

Ranking	Number of members		
	5	2	2
1st	O	S	S
2nd	N	N	O
3rd	S	O	N

(a) Using the Borda count method determine the number of points received by each location: 3 points for every first-place vote, 2 points for every second-place vote, and 1 point for every third-place vote.

(b) Under the Borda count method, which location is the winner?

6. The party committee is choosing the theme for the next party between beach party (B), decades (D), or sports (S). The preferences of the 17 voters are portrayed in the following table.

Ranking	Number of members			
	7	3	4	3
1st	B	D	S	D
2nd	D	B	B	S
3rd	S	S	D	B

(a) Using the Borda count method determine the number of points received by each theme: 3 points for every first-place vote, 2 points for every second-place vote, and 1 point for every third-place vote.

(b) Under the Borda count method, which theme is the winner?

7. Refer to the preference table in problem 3. Which location would be selected using the Borda count method?

8. Refer to the preference table in problem 4. Which form of advertisement would be selected using the Borda count method?

9. Three candidates are running for junior class president: Paul, Colleen, and Sherry. Voters were instructed to rank the three candidates in their order of preference. The results of the election appear in the following table.

Candidate	1st place votes	2nd place votes	3rd place votes
Paul	44	62	34
Colleen	76	35	29
Sherry	20	43	77

(a) Under the Borda count method, who is selected as junior class president?

(b) If the plurality method is used instead, do the results change? Why?

10. Another three candidates are running for senior class president: Hank, Roger, and Irene. Voters were instructed to rank the three candidates in their order of preference. The results of the election appear in the following table.

Candidate	1st place votes	2nd place votes	3rd place votes
Hank	39	30	38
Roger	33	21	53
Irene	35	56	16

(a) Who is chosen under the plurality method?

(b) Who is chosen under the Borda count method?

11. Refer to the preference table in problem 3. Determine the winning location by using the plurality with elimination method. After determining which candidate must be eliminated in each step, create a new preference table that does not include the eliminated candidate.

12. The hockey team is voting on the player who should receive the Rookie of the Year award. Their choices are Reed (R), Tom (T), or Jason (J). They have decided to use the pairwise comparison method to do so. The preferences of the 23 players are shown in the following preference table.

Ranking	Number of players					
	7	4	6	3	2	1
1st	R	T	T	J	J	R
2nd	T	R	J	T	R	J
3rd	J	J	R	R	T	T

(a) Using the pairwise comparison method there are three comparisons to consider. List the three comparisons and, for each comparison, indicate which option is preferred and by what margin.

(b) Calculate the number of points received by each candidate under the pairwise comparison method. Who is selected as winner of the award?

13. The entire Sauvageau family is voting on which type of food to have catered for their family reunion. The 8 family members have decided to use the pairwise comparison method to make their choice between Chinese (C), Italian (I), Greek (G), or Mexican (M). Their preferences appear in the following table.

Ranking	Grandma	Grandpa	Mom	Dad	Aunt	Son 1	Son 2	Daughter
1st	C	C	G	I	M	M	C	I
2nd	I	M	I	M	C	G	M	C
3rd	G	G	C	C	G	I	I	M
4th	M	I	M	G	I	C	G	G

(a) Using the pairwise comparison method there are six comparisons to consider. List the six comparisons and, for each comparison, indicate which option is preferred and by what margin.

(b) Calculate the number of points received by each option under the pairwise comparison method. Which type of food will be chosen for the party?

14. The 26 members of the Art Club are electing a new president. Their three choices are Jim, Ann, and Cal. Their preferences appear in the following table.

Ranking	Number of members				
	11	5	3	6	1
1st	Ann	Jim	Cal	Jim	Ann
2nd	Jim	Cal	Jim	Ann	Cal
3rd	Cal	Ann	Ann	Cal	Jim

(a) Under the plurality with elimination method, who is selected as president?

(b) Who is selected as president if Art Club chooses to use the pairwise comparison method instead?

15. Using the pairwise comparison method requires voters to make comparisons among all candidates. How many pairwise comparisons must be made with 6 candidates? What about with 9 candidates?

16. How many pairwise comparisons must be made with 5 candidates? How many with 8 candidates?

17. If there are three candidates in an election, how many different ways can they be ranked in order of preference? List the possible rankings.

18. If there are four candidates in an election, how many different ways can they be ranked in order of preference? List the possible rankings.

19. Suppose an election includes 4 candidates and 100 voters. Under the Borda count method, determine the total number of points possible.

20. Suppose an election includes 8 candidates and 200 voters. Under the Borda count method, determine the total number of points possible.

7.2 Flaws in the Voting Systems

In the last section we saw that the voting method used is just as significant as the voters' preferences in determining the winner. In this section, we will discuss **fairness criteria**, which are the properties that we expect a practical, balance voting system to fulfill. We will notice that, under some circumstances, the different voting systems can not always satisfy these standards.

The Majority Criterion

The **majority criterion** states that if one candidate is the first choice of the majority of voters, that candidate should be the winner. For example, if Stacey, John, and Erik are running for student body president and Erik is the first choice of 601 of the $1,200$ voters, then Erik should be selected.

However, notice that the majority criterion does not mention a case when no candidate receive the majority nor that the winner of an election must receive the majority of votes. We know that both the plurality method and the plurality with elimination method fulfill the majority criterion. This is because if a candidate has the majority of the votes, all other candidates have less than 50% of the votes and hence the candidate with the majority will win under the plurality method. In fact, if a candidate has a majority of the votes, all of the other candidates *combined* have less than 50% of the votes and so the candidate with the majority will win under the plurality with elimination method. However, the Borda count method, in which points are assigned to the candidates based on the voters' rankings, sometimes fails to satisfy the majority criterion as you will find in the next example.

Example 7.6. The Carnival Committee of the PTA, consisting of 9 members, is trying to decide where the annual school carnival should be held. The four choices are the school (S), the park (P), the Rec Center (R), or the gymnasium (G). Members were asked to rank the four choices first through fourth. The results are depicted in the following preference table. Using the results, determine (a) which location would be chosen by using a majority of first-place votes and (b) which location would be chosen by using the Borda count method.

Ranking	Number of members				
	3	2	2	1	1
1st	S	P	S	G	P
2nd	P	R	G	P	R
3rd	R	G	P	S	S
4th	G	S	R	R	G

Solution. (a) The following first-place votes, appearing in the first row of the preference table, were achieved for each site:

School: $3 + 2 = 5$ votes

Park: $2 + 1 = 3$ votes

Rec Center: 0 votes

Gymnasium: 1 vote

If determining the winning location by majority, the school wins with a total of 5 votes which is a majority of the nine votes casted.

(b) To use the Borda count method to determine the winner, we must figure out the number of first-, second-, third-, or fourth-place votes that were received for each location. Then we will assign points by granting 4 points for every first-place vote, 3 points for every second-place vote, 2 points for every third-place vote, and 1 point for every fourth-place vote.

School: $(5 \times 4) + (0 \times 3) + (2 \times 2) + (2 \times 1) = 26$ points

Park: $(3 \times 4) + (4 \times 3) + (2 \times 2) + (0 \times 1) = 28$ points

Rec Center: $(0 \times 4) + (3 \times 3) + (3 \times 2) + (3 \times 1) = 18$ points

Gymnasium: $(1 \times 4) + (2 \times 3) + (2 \times 2) + (4 \times 1) = 18$ points

Therefore, using the Borda count method, the park wins with a total of 28 points. However, we saw in (a) that the school won by a majority, therefore, in this case, the Borda count violates the majority criterion. ■

The Head-to-Head Criterion

It is reasonable to expect that the candidate who is the first choice of the majority of voters should be selected as the winner, which we explored in the majority criterion. It is also reasonable to expect that if a candidate is favored when paired separately with each of the other candidates, then that candidate should be selected. For example, if voters in an election for student body president involving Stacey, John, and Erik, prefer Stacey over John and they also prefer Stacey over Erik, then Stacey should be elected. This property is referred to as the **head-to-head criterion**.

Notice that the head-to-head criterion says nothing about the situation in which there is no such candidate that fits the property. The head-to-head criterion is sometimes called the **Condorcet criterion** and if there is a candidate who is favored in every comparison to the other candidates, this candidate is sometimes called the **Condorcet candidate**. In our previous example, Stacey would be the Condorcet candidate.

If the voting method used satisfies the head-to-head criterion, then the Condorcet candidate is selected. Also, if one candidate receives the majority of the votes, that candidate will also win any head-to-head comparison with the other candidates. Therefore, if a method satisfies the head-to-head criterion, it automatically satisfies the majority criterion also.

Example 7.7. Consider the following preference table that depicts the results of 9 voters' ballots in the election for chairperson of the advertising department. The three options were Callie (C), Alex (A), or Derek (D). Using the following preference table, complete the following:

(a) Under the plurality method, who would be elected?

(b) Show that this example violates the head-to-head criterion.

	Number of Ballots		
Ranking	4	3	2
1st	C	A	D
2nd	A	C	A
3rd	D	D	C

Solution. (a) Using the plurality method, we must determine who received the most first-place votes. By looking at the first row in the preference table, we can see that Callie received four first-place votes whereas Alex received 3 and Derek only received 2. Therefore, under the plurality method, Callie will be elected as the advertising department chair.

(b) The head-to-head criterion requires us to first determine which candidates are favored over the others. The first column shows us that 4 voters prefer C over A and the second and third columns show us that 5 voters prefer A over C. Thus, A is preferred over C. A is also preferred over D by 7 to 2. Thus, A is the Condorcet candidate since A is preferred over both C and D. This is a case in which the plurality method violates the head-to-head criterion. ■

The Monotonicity Criterion

A campaign usually precedes an election. A campaign is a time when voters learn more about each candidate, reflect on their options, and may change their preferences. Another reasonable expectation is that if a candidate gains support at the expense of his fellow candidates, that candidate should now have a better chance of winning the election. If a candidate were already in a position to be selected, increasing support of this candidate from the voters should definitely help his or her chances.

This thought process brings us to the monotonicity criterion. The **monotonicity criterion** states that if a candidate X is selected in an election and in a re-election the only changes to ballots favor candidate X (and only X) then candidate X should still be selected.

Example 7.8. The 11 members of Theatre Club are voting for their Vice-President with the choices being Darcy (D), Erica (E), or Justin (J). Their results are depicted in the following preference table:

	Number of Ballots			
Ranking	4	3	2	2
1st	D	E	E	J
2nd	E	J	D	D
3rd	J	D	J	E

(a) Determine who will be elected under the plurality with elimination method.

(b) Suppose both of the students who ranked the candidates in the order JDE changed their ballot to rank the candidates in the order DEJ. Would that affect the outcome of the election?

Solution. (a) To use the plurality with elimination method, we first need to calculate the number of first-place votes received by each candidate. Darcy received 4, Erica received 5, and Justin received 2. Since no candidate received a majority, Justin will be eliminated since he received the least. Now we produce a new preference table that does not include Justin.

	Number of Ballots	
Ranking	6	5
1st	D	E
2nd	E	D

Now Darcy has received 6 first-place votes which is a majority of the 11 votes casted. Thus, she will be elected as Vice-President of the Theatre Club under the plurality with elimination method.

(b) If two voters switched their preference ranking from

1st Justin

2nd Darcy

3rd Erica

in which Justin was one ranking above Darcy, the winner in part (a), to

1st Darcy

2nd Erica

3rd Justin

in which Darcy's position has been increased. It would seem reasonable, then, to expect Darcy to still win the election since the change of the ballot was in her favor. So, we will create a new preference table that includes this change in the ballots.

	Number of Ballots		
Ranking	6	3	2
1st	D	E	E
2nd	E	J	D
3rd	J	D	J

Now we will apply the plurality with elimination method to this table to see if the ballot change results in a different winner of the election. Darcy has now received 6 first-place votes, Erica received 5, and Justin did not receive any. Thus, Justin is again eliminated and we will re-create the preference table to not include Justin as a candidate.

	Number of Ballots	
Ranking	6	5
1st	D	E
2nd	E	D

Again, we calculate the number of first-place votes received by each candidate: Darcy received 6 and Erica received 5. Therefore, Darcy wins the election since she received a majority of the 11 first-place votes. This is compliant with part (a).

Thus, this is one example in which the plurality with elimination method satisfies the monotonicity criterion. If part (b) had produced a different winner than part (a), we would have said that it did not satisfy the monotonicity criterion. ■

The Irrelevant-Alternatives Criterion

Our last reasonable expectation of a fair election concerns what happens when a candidate is either added or removed at the last minute. If, in our previous example concerning the student body president election, John knows that Erik is favored to win and decides to drop out of the election, we would expect that Erik would still win. Also, if Ashley decides to run for student body president at the last minute but is not very popular to the voters, we would still expect Erik to win.

This is referred to as the **irrelevant-alternatives criterion** because John and Ashley are seen as irrelevant to the results of the election. Although this criterion seems very obvious, you will see in the next example that a fair election does not always fulfill the irrelevant-alternatives criterion.

Example 7.9. A history group of 8 students is deciding which topic to use for their semester project. They have three choices: Egyptian pyramids (E), Roman Empire (R), or the Great Wall of China (W). Under the plurality with elimination method,

(a) determine which topic will be chosen and

(b) determine if eliminating the Roman Empire as a topic, due to too little information, will effect the outcome of the election.

The students' preferences are shown in the following preference table.

	Number of Ballots		
Ranking	3	2	3
1st	E	W	R
2nd	R	E	W
3rd	W	R	E

Solution. (a) First we will determine the number of first-place votes received by each topic according to the above preference table. The Egyptian pyramids received 3 first-place votes, the Great Wall received 2, and the Roman Empire received 3. The Great Wall of China as a topic is eliminated and we re-create the preference table.

	Number of Ballots	
Ranking	5	3
1st	E	R
2nd	R	E

Now, the Egyptian pyramids received 5 first-place votes and the Roman Empire received 3. Thus, under the plurality with elimination method, the students will do their semester project on the Egyptian pyramids.

(b) If we remove the Roman Empire from the list of choices, we will only be considering the Egyptian pyramids and the Great Wall of China as options. According to the irrelevant-alternatives criterion, eliminating the Roman Empire should not affect the outcome since we know from part (a) that the Egyptian pyramids was the winning topic. If we eliminate R from the preference table we have:

Ranking	Number of Ballots	
	3	5
1st	E	W
2nd	W	E

Now we count the number of first-place votes received by each topic: Egyptian pyramids received 3 and the Great Wall of China received 5. Thus, by a majority of first-place votes, the Great Wall of China is now the winning topic. This is one example in which the plurality with elimination method does not satisfy the irrelevant-alternatives criterion. ■

Summary of the Fairness Criteria

In this section, we have seen that each of the four voting methods can sometimes infringe on one of the fairness criterion. The **Arrow's Impossibility Theorem** tells us that there is no voting method that satisfies all of the fairness criteria. The following table illustrates which voting methods satisfy which fairness criteria.

Method	Majority	Head to Head	Monotonicity	Irrelevant Alternatives
Plurality	Always	May Not	Always	May Not
Borda count	May Not	May Not	Always	May Not
Plurality with elimination	Always	May Not	May Not	May Not
Pairwise comparison	Always	Always	Always	May Not

Looking at the table, it may appear as though the pairwise comparison method is ideal since it satisfies three of the four fairness criteria. However, it does not always produce a winner. For example, consider the following preference table.

Ranking	8 votes	7 votes	6 votes
1st	B	C	A
2nd	A	B	C
3rd	C	A	B

You can see in the table that eight voters ranked B over A, seven voters ranked C over B, and six voters ranked A over C. So this preference table says that B is preferred to A, C is preferred to B, and A is preferred to C which does not provide a conclusion or a winner of the election.

Exercises

1. Consider the following preference table of 7 voters with 3 options:

Ranking	Number of Ballots		
	4	1	2
1st	A	C	B
2nd	B	B	C
3rd	C	A	A

   (a) Which candidate would win the election if the winner was required to receive a majority of the first-place votes?
   (b) Who is the winner of the election under the Borda count method?
   (c) Which criterion has been violated?

2. Consider the following preference table of 13 voters with three choices:

Ranking	Number of Ballots			
	7	3	2	1
1st	A	B	B	C
2nd	B	C	A	B
3rd	C	A	C	A

   (a) Which candidate would win the election if the winner was required to receive a majority of the first-place votes?
   (b) Who is the winner of the election under the Borda count method?
   (c) Which criterion has been violated?

3. Consider the following preference table portraying the preferences of 12 voters with three choices. Show how the plurality method violates the head-to-head criterion.

Ranking	Number of Voters		
	5	4	3
1st	B	A	C
2nd	A	B	A
3rd	C	C	B

4. Consider the following preferences table portraying the preferences of 10 voters with three choices. Show how the plurality method violates the head-to-head criterion.

Ranking	Number of Voters		
	4	3	3
1st	C	B	A
2nd	A	C	B
3rd	B	A	C

5. The city department, which includes 15 members, is voting on their new president with three choices: Candidate A, Candidate B, or Candidate C. The following preference table depicts their preferences.

Ranking	Number of Members				
	5	3	3	2	2
1st	A	B	B	A	C
2nd	B	A	C	C	A
3rd	C	C	A	B	B

   (a) Under the plurality method, which candidate is the winner?

   (b) List all pairwise comparisons and indicate who won and by what margin.

   (c) Do the results from parts (a) and (b) satisfy all the fairness criteria? If yes, why? If no, which criterion is violated?

6. The school board has four options, A, B, C, or D, for how to fit the required amount of school days in the upcoming year. The following table depicts the board members' preferences.

Ranking	Number of Members						
	4	2	1	1	1	1	1
1st	A	B	C	D	B	D	C
2nd	B	A	D	B	D	C	A
3rd	D	C	B	A	A	B	D
4th	C	D	A	C	C	A	B

   (a) Under the plurality method, which plan is the winner?

   (b) List all pairwise comparisons and indicate which plan won and by what margin.

   (c) Do the results from parts (a) and (b) satisfy all the fairness criteria. If yes, why? If no, which criterion is violated?

For Exercises 7-10 use the following table that depicts the preferences of 11 voters on 3 options.

Ranking	Number of Voters				
	4	3	2	1	1
1st	A	C	C	C	A
2nd	B	B	A	B	C
3rd	C	A	B	A	B

7. Who is the winner under the plurality method?

8. List all pairwise comparisons and indicate who won each and by what margin. Who is the winner under the pairwise comparison method?

9. Who is the winner under the Borda count method?

10. Consider your answers from questions 7, 8, and 9. List all fairness criteria that have been violated and explain why.

For Exercises 11-14 use the following table that depicts the preferences of 9 *voters on* 3 *options.*

Ranking	Number of Voters			
	3	3	2	1
1st	A	B	C	A
2nd	B	C	B	C
3rd	C	A	A	B

11. Who is the winner under the plurality method?

12. List all pairwise comparisons and indicate who won each and by what margin. Who is the winner under the pairwise comparison method?

13. Who is the winner under the Borda count method?

14. Consider your answers from questions 11, 12, and 13. List all fairness criteria that have been violated and explain why.

15. The following preference table contains the preferences of 51 voters with 4 choices.

Ranking	Number of Voters					
	7	9	3	18	10	4
1st	A	A	B	B	A	C
2nd	D	D	C	D	B	B
3rd	B	C	D	A	C	D
4th	C	B	A	C	D	A

    (a) Under the Borda count method, who is the winner?
    (b) Under the pairwise comparison method, who is the winner?
    (c) Under the plurality with elimination method, who is the winner?
    (d) Consider your answers to parts (a), (b), and (c). Identify which fairness criteria are violated and explain why.

16. The following preference table contains the preferences of 13 voters when faced with 3 choices.

Ranking	Number of Voters		
	6	4	3
1st	B	C	C
2nd	A	B	A
3rd	C	A	B

(a) Under the Borda count method, who is the winner?

(b) Under the pairwise comparison method, who is the winner?

(c) Under the plurality with elimination method, who is the winner?

(d) Consider your answers to parts (a), (b), and (c). Identify which fairness criteria are violated and explain why.

17. Consider the following preference table for 7 voters and 3 choices.

	Number of Voters		
Ranking	3	2	2
1st	B	C	B
2nd	C	A	A
3rd	A	B	C

(a) Who is the winner under the plurality method?

(b) Consider the situation in which the election is held a second time, only this time, Candidate C removed himself from the election. Who is the winner under the plurality method?

(c) Which of the fairness criteria has been violated?

18. Consider the following preference table for 9 voters and 3 choices.

	Number of Voters		
Ranking	4	3	2
1st	B	A	C
2nd	C	C	A
3rd	A	B	B

(a) Who is the winner under the plurality method?

(b) Consider the situation in which the election is held a second time, only this time, Candidate A removed himself from the election. Who is the winner under the plurality method?

(c) Which of the fairness criteria has been violated?

19. The following table depicts the preferences of 5 voters when faced with three options. Determine if the Borda count method violates the irrelevant-alternatives criterion.

	Number of Voters	
Ranking	2	3
1st	C	A
2nd	B	C
3rd	A	B

20. The following table depicts the preferences of 7 voters when faced with four options. Determine if the plurality with elimination method violates the head-to-head criterion.

	Number of Voters		
Ranking	1	3	3
1st	B	C	C
2nd	D	D	D
3rd	C	B	A
4th	A	A	B

7.3 Weighted Voting Systems

The four voting systems that we have been working with in the last two sections assume that all voters' have an equal voice in determining the outcome of the election. In this section, we will be looking at weighted voting systems in which equal voice is not the case. The presidential election is an example of a weighted voting system since the election is a vote by the electoral college. Each state receives a number of votes based on its number of U.S. senators plus its number of U.S. representatives. Major companies also use a weighted voting system when making major company decisions. Each stockholder receives a different number of votes based on the amount of stock they hold in the company. We will begin this section with some necessary vocabulary terms.

- **Weight:** It can be confusing to talk about each voter receiving more than one vote. So, instead, we talk about the **weight** assigned to each voter. For example, each state's weight in the electoral college is it's number of U.S. senators plus it's number of U.S. representatives. We will be assuming that the weights are all positive integers. We will let $P_1, P_2, \ldots, P_N$ be the voters where N is the number of voters since, for mathematical purposes, names are not important. We will let $W_1, W_2, \ldots, W_N$ be the corresponding weights of the voters. The voters and their weights can appear in a table:

Voter	Weight
P_1	11
P_2	9
P_4	5
P_3	7

The entries in the table can also be written in the following notation using brackets: $[11, 9, 7, 5]$ in descending order by weight. Note the order in the table is not necessarily the order used in the brackets. Using the customary variables, this table tells us that $W_1 = 11$, $W_2 = 9$, $W_3 = 7$, and $W_4 = 5$.

- **Motions:** We will limit our discussion in this section to deal only with yes/no questions. These simple yes/no votes are sometimes referred to as **motions**. A final decision of "yes" **passes** the motion while a final decision of "no" **defeats** the motion.

- **Simple Majority:** A **simple majority** refers to the requirement that a candidate must receive more than half the votes in order to be declared the winner.

- **Supermajority:** Sometimes the requirement is even higher than more than half the votes. This is referred to as a **supermajority** if the election requires the winner to receive more than a simple majority. For example, an election may require a candidate to receive $\frac{2}{3}$ of the votes.

- **Quota:** A **quota** is the minimum number of votes needed to pass a motion. For example, the weighted voting system depicted in the above table has a total weight of
$$11 + 9 + 7 + 5 = 32$$
So half the total weight is $\frac{32}{2} = 16$. So the quota will have to be above 16 and since we are only using whole numbers, the quota would be 17. The quota can be written as part of the above notation that is used for listing the weights. The weights and quota from the above example would be represented by: $[17|11, 9, 7, 5]$.

Example 7.10. Consider the weighted voting system $[21|13, 8, 7, 4]$. Suppose that voters P_1 and P_3 voted yes on a motion. Did the motion pass?

Solution. The total weight in this voting system is

$$13 + 8 + 7 + 4 = 32$$

So the quota, 21, is 5 over half of the total weight, 16. The sum of the weights of the voters who voted yes is $W_1 + W_3 = 13 + 7 = 20$. Since 20 is less than the quota of 21, the motion is defeated. ■

Example 7.11. Consider the weighted voting system $[25|15, 11, 8, 5]$. Suppose that voters P_1, P_3, and P_4 voted yes on a motion. Was the motion passed or defeated?

Solution. The total weight in this voting system is

$$15 + 11 + 8 + 5 = 39$$

The sum of the weights of the voters who voted yes is

$$W_1 + W_3 + W_4 = 15 + 8 + 5 = 28$$

Since 28 is greater than the quota of 25, the motion passes. ■

It is important that the quota is not too large or too small. Let's look at the weighted voting system used in Example 7.10. If the quota is lowered to 10, it is not allowing any type of decision making. Imagine if P_1 and P_3 vote yes $(13 + 7 = 20)$ and P_2 and P_4 vote no $(8+4 = 12)$. In this case, we have a stalemate because both the Yes's and the No's have enough votes to pass the quota and win. Now imagine if the quota is raised to 34. Given that there are only 32 votes in the system even if every voter were to vote yes, the motion would always be defeated. We call this a *gridlock*.

Coalitions

A **coalition** is a nonempty set of voters in a weighted voting system. It can consist of any number of voters, however the number of possible coalitions is dependent on the number of voters in the weighted voting system. The number of possible coalitions in a weighted voting system with n voters is $2^n - 1$.

Example 7.12. In a weighted voting system of 5 voters, how many coalitions are possible?

Solution. Using the formula, the number of possible coalitions is

$$2^5 - 1 = 32 - 1 = 31$$

Thus, there are 31 possible coalitions. ■

Example 7.13. Consider the weighted system appearing in the following table containing the weights of voters A, B, C, D, E, and F.

Voters	Number of Votes
A	18
B	11
C	4
D	15
E	7
F	7

(a) The election retains the requirement that a motion must receive 75% of the votes in order to be passed. What is the quota for this weighted voting system?

(b) How many coalitions are possible in this weighted voting system?

Solution. (a) The total number of votes in this voting system is

$$18 + 11 + 4 + 15 + 7 + 7 = 62$$

Since a motion must receive 75% of the votes in order to be passed, the quota is 75% of the total number of votes. Thus, the quota is

$$0.75 \times 62 = 46.5 \text{ which we will round up to } 47 \text{ votes.}$$

The quota for this weighted voting system is 47 votes.

(b) Using the formula with $n = 6$, we know that there are $2^6 - 1 = 63$ possible coalitions in this weighted voting system. ■

A coalition can be either a winning coalition or a losing coalition depending on the relationship between the total weight of the voters and the quota in the voting system. A coalition is a **winning coalition** if the total weight of the voters is greater than or equal to the quota. On the other hand, a coalition is a **losing coalition** if the total weight of the voters is less than the quota. We will be using set notation to list all possible coalitions.

Example 7.14. Consider the weighted voting system $[7|5, 4, 3]$. List all possible coalitions and determine whether each is a winning or a losing coalition.

Solution. The coalitions with one voter are $\{P_1\}, \{P_2\}, \text{and}\{P_3\}$. The coalitions with two voters are $\{P_1, P_2\}, \{P_2, P_3\}, \text{and}\{P_1, P_3\}$. Lastly, there is one coalition with all three voters, $\{P_1, P_2, P_3\}$. We will list all the coalitions in the following table and determine the total weight of each. Any coalition with a total weight greater than or equal to 7 is a winning coalition and all others are losing coalitions.

Coalition	Total Weight	Winning or Losing
$\{P_1\}$	5	Losing
$\{P_2\}$	4	Losing
$\{P_3\}$	3	Losing
$\{P_1, P_2\}$	$5 + 4 = 9$	Winning
$\{P_2, P_3\}$	$4 + 3 = 7$	Winning
$\{P_1, P_3\}$	$5 + 3 = 8$	Winning
$\{P_1, P_2, P_3\}$	$5 + 4 + 3 = 12$	Winning

■

Critical Voters

A **critical voter** is a voter whose weight is large enough to change a winning coalition to a losing coalition by leaving that coalition. Thus, if a voter leaves the winning coalition and causes the total weight of the coalition to drop so that the coalition is now a losing coalition, that voter is referred to as the critical voter since their vote is vital to the coalition's success.

Dictators, Dummies, and the Power of the Veto

It seems as though a voter's weight depicts their power, but this is not always the case. A **dummy** is a voter without power. They have no influence on the outcome of a vote. A **dictator** is a voter whose weight is greater than or equal to the quota and, therefore, has absolute power. It does not matter how the other voters vote, the dictator has enough votes to pass or defeat a motion on their own. If a weighted voting system has a dictator, the other voters are automatically dummies. A voter has **veto power** if he has the power to prevent any motion from passing. These terms will be more clearly illustrated in the following examples.

Example 7.15. Consider the following weighted voting system: $[16|10, 6, 5]$.

(a) List all the coalitions and determine whether each is a winning or a losing coalition.

(b) List all coalitions containing P_3. Then remove P_3 from each and determine the effect of having P_3 in a coalition.

(c) Identify which voters are the dummies and which are the dictators.

Solution. (a) The following table lists all the coalitions, their total weight, and whether they are winning or losing.

Coalition	Total Weight	Winning or Losing
$\{P_1\}$	10	Losing
$\{P_2\}$	6	Losing
$\{P_3\}$	5	Losing
$\{P_1, P_2\}$	$10 + 6 = 16$	Winning
$\{P_2, P_3\}$	$6 + 5 = 11$	Losing
$\{P_1, P_3\}$	$10 + 5 = 15$	Losing
$\{P_1, P_2, P_3\}$	$10 + 6 + 5 = 21$	Winning

(b) The following table lists all coalitions containing P_3 on the left side. The right side contains these coalitions after removing P_3 from the coalition on the left.

Coalitions with P_3	Coalitions without P_3
$\{P_3\}$ Losing	No voters – Not a coalition
$\{P_2, P_3\}$ Losing	$\{P_2\}$ Losing
$\{P_1, P_3\}$ Losing	$\{P_1\}$ Losing
$\{P_1, P_2, P_3\}$ Winning	$\{P_1, P_2\}$ Winning

Notice that removing P_3 from each of the coalitions does not affect whether the coalition is winning or losing. Thus, it does not matter if P_3 is in a coalition or not.

(c) Since P_3 does not have any power in the election, as we saw in part (b), we saw that P_3 is the dummy. We can also show that neither P_1 nor P_2 are dummies. There is no dictator in this example since no voter has a weight greater than or equal to the quota. ■

The Banzhaf Power Index

The **Banzhaf power** of a voter in a weighted voting system is defined as the number of winning coalitions in which that voter is a critical voter. The **total Banzhaf power** in the weighted voting system is the sum of the Banzhaf powers of all voters in a voting system. Each individual voter has a **Banzhaf power index** —the ratio of the voter's Banzhaf power to the total Banzhaf power in the weighted voting system.

> **To Calculate Voters' Banzhaf Power Indices:**
>
> **STEP 1:** Locate all winning coalitions in the weighted voting system.
>
> **STEP 2:** Determine which voters are critical in each of the winning coalitions.
>
> **STEP 3:** Calculate each voter's Banzhaf power by adding up the number of times each voter is critical.
>
> **STEP 4:** Calculate the total Banzhaf power for the weighted voting system by adding all voters' Banzhaf powers.
>
> **STEP 5:** Determine each voter's Banzhaf power index by dividing his/her Banzhaf power by the total Banzhaf power.

Example 7.16. Given the weighted voting system from Example 7.15: $[16|10,6,5]$. Calculate the Banzhaf power index for each voter.

Solution. STEP 1: We need to list all coalitions in the system, calculate their total weight, and determine whether they are a winning or a losing coalition. The results are in the following table:

Coalition	Total Weight	Winning or Losing
$\{P_1\}$	10	Losing
$\{P_2\}$	6	Losing
$\{P_3\}$	5	Losing
$\{P_1, P_2\}$	$10+6=16$	Winning
$\{P_2, P_3\}$	$6+5=11$	Losing
$\{P_1, P_3\}$	$10+5=15$	Losing
$\{P_1, P_2, P_3\}$	$10+7+5=22$	Winning

STEP 2: Next we need to determine which voters are critical in each coalition. Our weighted voting system only has 2 winning coalitions, therefore, we need to remove each voter one at a time from each winning coalition to determine if the winning coalition turns into a losing coalition. If it does change, the voter removed from the coalition is a critical voter in the original coalition. If it does not change, the removed voter is not a critical voter. These results are in the following table:

Original Coalition	New Coalition	Weight of New Coalition	Still a Winning Coalition?	Critical Voter
$\{P_1, P_2\}$	$\{P_1\}$	10	No	P_2
$\{P_1, P_2\}$	$\{P_2\}$	6	No	P_1
$\{P_1, P_2, P_3\}$	$\{P_1, P_2\}$	16	Yes	
$\{P_1, P_2, P_3\}$	$\{P_2, P_3\}$	11	No	P_1
$\{P_1, P_2, P_3\}$	$\{P_1, P_3\}$	15	No	P_2

STEP 3: Next we need to count the number of times each voter is a critical voter. According to the table, P_1 and P_2 are both critical voters 2 times.

STEP 4: Now we need to add up all the voters' Banzhaf powers to calculate the total Banzhaf power in the voting system. The total Banzhaf power in our voting system is 4 since P_1 and P_2 are both critical voters 2 times.

STEP 5: Now we need to calculate each voter's Banzhaf power index by dividing each Banzhaf power by the total Banzhaf power. Thus,

$$P_1 : \text{Banzhaf power index} = \frac{2}{4} = \frac{1}{2} = 50\%$$

$$P_2 : \text{Banzhaf power index} = \frac{2}{4} = \frac{1}{2} = 50\%$$

Notice that the sum of the voters' Banzhaf power indices equal 1, or 100%. It is not always the case that the Banzhaf power indices will be equal. ■

Exercises

1. Voters A, B, C, and D have weights $4,\ 7,\ 3,$ and 9, respectively.
   (a) What is the smallest weight required to pass a motion with the requirement that the motion receive a simple majority of the votes?
   (b) What is the smallest weight required to pass a motion with the requirement that the motion receive $\frac{2}{3}$ of the votes?
   (c) With a quota of 11, express the weighted voting system using the correct notation.
2. Voters Ann, George, Paul, and Rita have weights $11,\ 9,\ 4,$ and 8, respectively.
   (a) What is the smallest weight required to pass a motion with the requirement that the motion receive a simple majority of the votes?
   (b) What is the smallest weight required to pass a motion with the requirement that the motion receive $\frac{2}{3}$ of the votes?
   (c) With a quota of 14, express the weighted voting system using the correct notation.

For Exercises 3-8 a simple majority is required to pass a motion. Calculate the quota for each and use proper notation to express the weighted voting system.

3. $[13, 9, 8, 7, 7, 4, 2, 2, 1]$
4. $[6, 6, 5, 3, 3, 3, 2, 2]$

5. $[18, 14, 12, 12, 9, 5, 3, 3]$
6. $[8, 6, 6, 5, 4, 2, 1]$
7. $[5, 5, 3, 2, 2, 2]$
8. $[21, 18, 13, 13, 9, 5, 2, 2]$
9. For the given weighted voting systems, calculate the quota if more than the given percentage is required for the motion to pass.
   (a) $[2, 1, 1, 1, 1]$; 60%
   (b) $[8, 5, 3, 3, 2, 1]$; 60%
   (c) $[10, 8, 6, 6, 4, 2]$; 75%
   (d) $[6, 5, 3, 3, 3, 2]$; 60%
10. Consider the weighted voting system $[12|5, 4, 3, 2]$.
   (a) Suppose P_1, P_2, and P_4 vote yes which P_3 votes no. Will the motion be passed or defeated?
   (b) Suppose P_1 and P_3 vote yes on a motion while P_2 and P_4 vote no. Will the motion pass or be defeated?
   (c) What must happen in order for a motion to pass?

For Exercises 11-14 determine if the coalitions given in parts (a) through (c) are winning or losing coalitions.

11. $[25|14, 13, 12, 8]$
   (a) P_1 and P_2
   (b) P_4 and P_3
   (c) P_2 and P_3
12. $[12|8, 5, 3, 3, 2, 1]$
   (a) P_1,P_2, and P_4
   (b) P_2,P_3 and P_6
   (c) P_3,P_4, P_5, and P_6
13. $[15|8, 4, 4, 3, 2, 1]$
   (a) P_1,P_2, and P_4
   (b) P_4 and P_3
   (c) P_2,P_3, and P_4
14. $[6|4, 3, 2, 1]$
   (a) P_1,P_2, and P_4
   (b) P_4 and P_3
   (c) P_2,P_3, and P_4

For Exercises 15-16 create a table containing all possible coalitions for the given weighted voting system.

15. (a) $[7|5, 4, 2]$
   (b) $[8|5, 3, 2, 1]$
16. (a) $[9|8, 5, 5, 2]$
   (b) $[4|2, 1, 1, 1, 1]$
17. Consider the weighted voting system $[17|11, 6, 3]$
   (a) What fraction of the weight does each voter have?
   (b) Determine the Banzhaf power index for each voter.
   (c) Compare your answers to parts (a) and (b). Explain why the weight controlled by each voter is not a good way to measure the voter's power.

18. Consider the weighted voting system $[24|22, 4, 2]$

    (a) What fraction of the weight does each voter have?

    (b) Determine the Banzhaf power index for each voter.

    (c) Compare your answers to parts (a) and (b). Explain why the weight controlled by each voter is not a good way to measure the voter's power.

For Exercises 19-20 calculate the Banzhaf power index for each voter. Also determine who are the dictators and who are the dummies.

19. (a) $[5|4, 1, 1, 1]$

    (b) $[8|6, 4, 2]$

20. (a) $[20|11, 10, 9]$

    (b) $[6|6, 2, 1, 1, 1]$

8

Fair Division

Fair division, the mathematics of sharing, is something that we begin learn early in life. You may have dealt with a situation like this: You and three friends are hanging out at your house and there are 40 M&M's. How do you divide them up so that everyone gets an even amount of candy? In this chapter, we will be looking at the different methods of fair division and situations in which the methods may be used.

8.1 Divide and Choose Methods

We will begin this section by introducing some necessary vocabulary terms and definitions. Then we will begin to look at the different methods of dividing and choosing.

Fair Division Definitions

This chapter focuses on the **fair-division problem** – how to divide a desired object up evenly. The parties who are wanting to share that desirable object are referred to as the **players**. The discovered solution to the fair-division problem is called the **fair-division procedure**. We will begin by discussing the three different types of fair-division problems.

- **Continuous Fair-Division Problems:** In this case, the desired object(s) is divisible infinitely many ways and shares can be increased or decreased in very small amounts that do not result in any loss of value. An example of this type of fair-division problem is the dividing of a cake.
- **Discrete Fair-Division Problems:** In this case, the desired object(s) is something that is indivisible, such as houses, cars, artworks, etc. Sometimes, the solution to this type of problem involves the addition of money in order to divide the object(s) fairly.

- **Mixed Fair-Division Problems:** In this case, some of the desired object(s) can be subdivided while others are not divisible. It is a combination of continuous and discrete fair-division problems. An example would be the dividing or sharing of an estate containing a home, land, jewelry, and artwork.

This first section will focus on continuous fair-division problems. We will discuss the other types in the following sections. We will now look at some general rules and assumptions that are part of fair-division problems.

Every person is different and, therefore, values different things. You may value your lake home while another person views it as worthless. This must be recognized by all fair-division problems such that *the value of a player's share is determined by his or her own preferences or values*.

Consider the following example:

> Jennifer and Rob are sharing a pizza that is half-pepperoni and half-cheese. They decide that it is only fair that they each get half of the pizza. We say that they have each received a fair part of the pizza if they each value the half that they have received. However, if Jennifer receives the cheese half of the pizza but loves pepperoni she does not view the division as fair. If Rob then received the pepperoni half of the pizza and also loves pepperoni, he may feel as though his half is much better than Jennifer's.

Thus, we have created the following definition:

- A player has received a **fair share** if he or she considers their share to be worth at least $\frac{1}{n}$ of the total value being shared, where n is the number of players in the fair division problem.
- If every player receives a fair share, we call the division **proportional**.

For example, in a situation involving 5 players, a player has received a fair share if his or her share is at least $\frac{1}{5}$ of the value being shared.

Fair-division rules do not allow a player to change his or her values after the division has been made. Thus, *a player's values in a fair-division problem cannot change based on the results of the division*. It is also necessary that *no player has knowledge of any other player's values*. This means that player A is not permitted to take advantage of knowing player B's preferences.

We will now begin to look at the applications of fair division. The remainder of the section is organized based on the number of players involved in the fair-division problem.

Fair Division Involving Two Players: The Divider-Chooser Method

A continuous fair-division problem involving only two players is the simplest to solve. It can sometimes be referred to as the "I cut, you choose" method. We will be using a basic pizza for our explanations. One person cuts the pizza into two pieces that they view as equal. Then, the second person gets to choose which piece they want and the remaining piece goes to the cutter. This divide-and-choose method is broken down into the following steps:

Divide-And-Choose Method For Two Players (Divider-Chooser): Two players, A and B, are sharing a pizza.

> **STEP 1:** Player A cuts the pizza into two pieces that he/she views as equal value. Thus, A is called the **divider**.
>
> **STEP 2:** Player B chooses one of the pieces that he/she views as the greatest value. If player B views both pieces as equal, he or she can choose either piece. Thus, B is called the **chooser**.
>
> **STEP 3:** Player A then receives the piece that was not chosen by player B.

Notice that this divide-and-choose method for two players is, by definition, proportional. Player B choose the piece that he considers to have the greatest or equal value so he will feel that his piece is preferable. Then player A receives the piece that is leftover, but since she cut the pizza in a way that she felt both pieces to be equal, she will also feel that her piece is preferable. Therefore, both players are happy with the equal results. We will now look at an example in which not all parts of the pizza are equally valuable to both players but the divide-and-choose method can still be used.

Example 8.1. Jonathon and Cheryl decided to eat lunch at Cheryl's house. Cheryl puts a $\$4$ pizza in the oven that is half-pepperoni and half-sausage. Cheryl likes both kinds of pizza equally but Jonathon likes pepperoni 4 times as much as sausage. That is, he would pay four times as much for a slice of pepperoni as he would for a slice of sausage. Cheryl cuts the cooked pizza into six equal slices and arranges them on two plates. The first plate contains two slices of sausage and one pepperoni and the second plate contains two slices of pepperoni and one sausage. What monetary value would Jonathon and Cheryl put on the original halves of the pizza? What value would each person place on each plate of pizza? Which plate would Jonathon choose?

Solution. Cheryl is the divider in this example. Because she likes both kinds of pizza equally well, she would assign half the value of the pizza to one kind of pizza and half the value to the other kind. So, for Cheryl, the pepperoni half of the pizza is worth $\$2$ and the sausage half of the pizza is worth $\$2$. To her, each plate of pizza is of equal value since each plate holds one-half of the pizza.

Jonathon is the chooser in this example and he prefers pepperoni to sausage 4 to 1. To him, the pepperoni half of the pizza is four times as valuable as the sausage half. Jonathon would assign $\frac{4}{5}$ of the value to the pepperoni half and $\frac{1}{5}$ of the value to the sausage half. Thus, the pepperoni half is worth

$$\frac{4}{5}(\$4) = \$3.20$$

and the sausage half is worth

$$\frac{1}{5}(\$4) = \$0.80$$

After Cheryl divides the pizza among the two plates, Jonathon will choose the plate that has the most value to him. Jonathon values each slice of pepperoni pizza at

$$\$3.20 \div 3 \approx \$1.067$$

and each slice of sausage pizza at

$$\$0.80 \div 3 \approx \$0.267$$

Thus, in Jonathon's value system, the plate containing two slices of pepperoni and one sausage is worth

$$2(1.067) + 1(0.267) \approx \$2.40$$

The plate containing one slice of pepperoni and two sausage is worth

$$1(1.067) + 2(0.267) \approx \$1.60$$

Thus, Jonathon will choose the plate with two slices of pepperoni and one slice of sausage which, to him, is worth more than half of the total value of the pizza. Cheryl is then left with the second plate of pizza which, to her, is worth half the total value of the pizza. ■

The pizza in the previous example could have been divided in a different way that would have satisfied both players. Jonathon would have been able to choose a plate more valuable to him had all the pepperoni pieces been on one plate and all the sausage pieces on another plate and Cheryl still would have received a plate that she valued. Our divide-and-choose method for two players does guarantee proportionality but it does not necessarily guarantee that all players will be completely satisfied.

The following example shows how we can assign points to the "cake" being divided when there is no way of determining value in terms of money.

Example 8.2. Parker and Isaac are borrowing a sports car to drive to a city that is 10 hours away. They want to fairly share the 6 hours of daylight driving and 4 hours of driving at night. Assume that Parker prefers night driving 2 to 1 over day driving. Isaac considers both times of driving equally fun. How should they divide the driving hours into two shifts if Parker is the divider and Isaac is the chooser?

Solution. In this example, the "cake" is the driving time, and the two "pieces" are the two shifts. We will use points in order to assign numerical values. Since Parker prefers night driving over day driving 2 to 1, he might assign every hour of night driving two points and every hour of day driving 1 point. So, he would give a total value of

$$(2 \times 4) + (1 \times 6) = 14 \text{ points}$$

to the total complete 10-hour drive. Since Parker is the divider, he would consider a fair division of the pieces to be $\dfrac{14}{2} = 7$ points for each driving shift.

Parker would now need to find a way to divide the driving hours into two equal shifts, 7 points each. One way to do this would be to allow the first shift to consist of 6 daytime hours and half an hour of nighttime driving;

$$(1 \times 6) + (2 \times 0.5) = 7 \text{ points}$$

Then the second shift would consist of 3.5 hours of nighttime driving;

$$2 \times 3.5 = 7 \text{ points}$$

Now that Caleb has made a division into two shifts that he views as equal, Isaac will choose the shift that he prefers.

In order to make his decision, Isaac will need to view the driving hours in terms of his own value system in which all hours have equal value. So we will say that he assigns 1 point per hour. Isaac would assign the first shift a value of

$$(1 \times 6) + (1 \times 0.5) = 6.5 \text{ points}$$

and he would assign the second shift a value of

$$1 \times 3.5 = 3.5 \text{ points}$$

Thus, Isaac would take the first 6.5 hour shift since it is of greater value to him than the second shift. Parker would then be happy with driving the second shift since he viewed both shifts to be of equal value. ■

Let's consider how the results from Example 8.2 would have differed had Isaac been the divider and Parker been the chooser. Isaac would have assigned 1 point for each of the 6 hours of daytime driving and one point for each of the 4 hours of nighttime driving. So, the total point value for all the driving hours, in Isaac's value system, would have been

$$(1 \times 6) + (1 \times 4) = 10 \text{ points}$$

A fair division would equal $\dfrac{10}{2} = 5$ points for each driving shift. One way to make the division would have been to allow the first shift to consist of 5 daytime hours and the second shift to consist of 1 daytime hour and 4 nighttime hours. In this case, Parker would have chosen the second shift since it has a total value of 9 points in his value system. Thus, Isaac would have driven the first 5 hours.

Both solutions allow the chooser to select a shift that is worth more than half the point value in their own eyes. Also, the divider ended up with a shift that is worth the same as the shift selected by the chooser. It may be best to randomly select who is the divider and who is the chooser so that both players view the method as completely fair.

Fair Division Involving Three Players: The Lone-Divider Method

Next we will look at a situation in which three players are required to divide and share something. The *lone-divider method* allows one person to be the divider and the other two players to play the role of the choosers. This method is similar to the divide-and-choose method previously discussed; however, it involves another step in which the choosers must state which pieces they believe are acceptable. The steps are described below.

Divide-And-Choose Method For Three Players (Lone-Divider): Three players, A, B, and C, are to divide a cake:

> **STEP 1:** Player A (divider) divides the cake into three pieces that he or she views as equal.
>
> **STEP 2:** Players B and C (choosers) individually decide which pieces are worth at least $\frac{1}{3}$ of the cake's value. These pieces are said to be **acceptable** to the players.
>
> **STEP 3:** The choosers announce which pieces they view as acceptable.
>
> **STEP 4:** This next step depends on which pieces the chooser considers acceptable.
>
> > **Case A:** If there is one piece that is unacceptable to either player B or player C, then player A, the divider, gets that piece. If it is possible for players B and C to choose different acceptable pieces from the remaining two choices, they do so. If not, they need to put the remaining two pieces back together and use the divide-and-choose method for two players to share the reassembled pieces.
> >
> > **Case B:** If every piece is viewed as acceptable by players B and C, then both B and C take pieces they view as acceptable. The divider gets the leftover piece.

Note that in step 2 of the method, not all the pieces can be worth less than one-third of the total. So, players B and C must find at least one piece that they view as acceptable. They may, however, view different pieces as acceptable. Note that in step 4, the piece given to player A is worth less than one-third of the cake's value, according to the values of players B and C. Thus, the two remaining pieces must have a value that is greater than two-thirds of the cake's total value.

Example 8.3. Suppose that Julia, Rachel, and Meredith want to fairly divide 24 cookies: eight vanilla, eight chocolate, and eight mixed. Julia likes all three flavors equally well so her preference ratio is 1 to 1 to 1. Rachel prefers chocolate 2 to 1 over vanilla and mixed, and she likes mixed and vanilla equally well. So her preference ratio of vanilla to chocolate to mixed is 1 to 2 to 1. Meredith's preference ratio is 1 to 2 to 3 because she likes chocolate twice as much as vanilla and mixed three times as much as vanilla. With Julia as the divider, apply the divide-and-choose method for three players to determine the result.

Solution. The "cake" in this example is the 24 cookies that need to be shared equally among the three players. We will use the divide-and-choose method for three players to see what kind of equal division results with Julia as the divider.

STEP 1: Julia divides the cookies into three equal portions. Since she does not prefer any flavor over the others, suppose that she divides the cookies into three groups, each consisting of one of the flavors. So we have one portion of 8 vanilla cookies, one portion of 8 chocolate cookies, and one portion of 8 mixed cookies. Using points to assign numerical values, Julia may assign 1 point to each cookie for a total of 24 points since she views all flavors equally. She would then assign 8 points to each of the 3 portions. Julia's values are shown in the following table.

Julia's Values

	Points per Cookie	Number of Cookies	Value
Portion 1 (Vanilla)	1	8	$1 \times 8 = 8$ points
Portion 2 (Chocolate)	1	8	$1 \times 8 = 8$ points
Portion 3 (Mixed)	1	8	$1 \times 8 = 8$ points

STEP 2: The next step requires the choosers to determine which pieces are acceptable. According to Rachel's preference ratio of 1 to 2 to 1, she may assign 1 point for every vanilla cookie, 2 points for every chocolate cookie, and 1 point for every mixed cookie. Her value's are shown in the following table.

Rachel's Values

	Points per Cookie	Number of Cookies	Value
Portion 1 (Vanilla)	1	8	$1 \times 8 = 8$ points
Portion 2 (Chocolate)	2	8	$2 \times 8 = 16$ points
Portion 3 (Mixed)	1	8	$1 \times 8 = 8$ points

Rachel's total point value for all the cookies is

$$8 + 16 + 8 = 32 \text{ points}$$

Under Rachel's value system, a fair share would be worth

$$\frac{1}{3} \times 32 = 10\frac{2}{3} \text{ points}$$

According to these values, Rachel would only view portion 2 as a fair share because it is the only portion that she views as worth $\frac{1}{3}$ of the total point value. Thus, Rachel finds only portion 2 as acceptable and views portions 1 and 3 as unacceptable.

According to Meredith's preference ratio of 1 to 2 to 3, she might assign 1 point for every vanilla cookie, 2 points for every chocolate cookie, and 3 points for every mixed cookie. Meredith's values are shown in the following table.

Meredith's Values

	Points per Cookie	Number of Cookies	Value
Portion 1 (Vanilla)	1	8	$1 \times 8 = 8$ points
Portion 2 (Chocolate)	2	8	$2 \times 8 = 16$ points
Portion 3 (Mixed)	3	8	$3 \times 8 = 24$ points

Meredith's total point value for all the cookies is

$$8 + 16 + 24 = 48 \text{ points}$$

According to her value system, Meredith would view a fair share as

$$\frac{1}{3} \times 48 = 16 \text{ points}$$

Looking at the table of Meredith's values shows us that both portions 2 and 3 would be viewed as an acceptable share by Meredith and portion 1 is unacceptable.

STEP 3: The choosers state which portions they view as acceptable. Rachel only views portion 2 as acceptable and Meredith views both portions 2 and 3 as acceptable.

STEP 4: Once choosers declare which pieces they view as acceptable, pieces are assigned to each player. Since both choosers view portion 1 as unacceptable, Julia, the divider, received portion 1. Rachel only views portion 2 as acceptable so she receives portion 2. Since portion 3 is the only piece left and Meredith viewed it as acceptable, she will receive portion 3. The final fair division results are as follows:

Julia: Portion 1 (Vanilla)

Rachel: Portion 2 (Chocolate)

Meredith: Portion 3 (Mixed)

■

Fair Division Involving Three or More Players: The Last-Diminisher Method

The divide-and-choose method can be extended to more than three players. In this case, we will use the *last-diminisher method* to divide and share the "cake". The steps of the method are described below.

The Last-Diminisher Method of Fair Division: Suppose any number of players A,B,... are dividing a cake. The following steps are used to give one player a piece of cake that he or she considers a fair share and that no other player considers to be more than fair.

STEP 1: Player A cuts a piece of the cake that he or she considers a fair share.

STEP 2: Each remaining player judges the fairness of that piece of cake.

Case A: If a player considers the piece to be a fair share or less than a fair share, the judging moves on to the next player.

Case B: If a player considers the piece to be larger than a fair share, that player must trim the piece to a smaller size that he or she considers fair. The piece that was trimmed off is reattached to the cake and it is the next player's turn to judge the piece.

STEP 3: The last player who trimmed the piece to a smaller size gets that piece of cake. If no player trimmed the piece, player A, who originally cut the piece, get the piece of cake.

Repeat: After one player receives a piece of the cake, repeat the entire process without that player and that piece. When only two players remain, use the divide-and-choose method for two players to divide the remaining cake fairly.

Example 8.4. Refer to Example 8.3: Julia, Rachel, and Meredith are, again, trying to fairly divide 24 cookies made up of 8 vanilla, 8 chocolate, and 8 mixed. Their preferences are still the same: Julia has a preference ratio of 1 to 1 to 1, Rachel has a preference ratio of 1 to 2 to 1, and Meredith has a preference ratio of 1 to 2 to 3. However, this time they use the last-diminisher method of fair division to share the cookies fairly. Assume that Meredith is serving the first portion and Julia is taking the first turn in judging the fairness of that portion.

Solution. Again, the "cake" in this example is the 36 cookies and the "piece of cake" being considered is the portion being served by Meredith.

STEP 1: Again we begin by granting points to assign numerical values. Meredith would assign 1 point for every vanilla cookie, 2 points for every chocolate cookie, and 3 points for every mixed cookie.

As in Example 8.3, Meredith's total point value for all the cookies is 48 points. In her value system, a fair share would be worth 16 points. So Meredith decides to make the first portion contain all 8 vanilla cookies and 4 of the chocolate cookies to reach a serving value of

$$(1 \times 8) + (2 \times 4) = 16 \text{ points}$$

There are other ways in which Meredith could have put together a fair share, but we will be using this serving in our example.

STEP 2: Now the other players, Julia and Rachel, must judge the portion put together by Meredith. Julia will judge first. Because she likes all the flavors equally well, she will assign 1 point for every cookie, no matter what the flavor.

As in Example 8.3, Julia's total point value for all the cookies is 24 points. In her value system, a fair share is worth 8 points. According to this value system, the serving put together by Meredith is worth 12 points since it contains 8 vanilla cookies and 4 chocolate cookies. Thus, in Julia's eyes, the serving is too large to be a fair share so she removes the 4 chocolate cookies to make the serving only consist of 8 vanilla cookies and, therefore, equaling 8 points. Note that this is only one way that Julia could have trimmed down the serving.

It is now Rachel's turn to judge the fairness of the serving left over after Julia's judgement. According to Rachel's value system, she would assign 1 point for every vanilla cookie, 2 points for every chocolate cookie, and 1 point for every mixed cookie.

As in Example 8.3, Rachel's total point value for all the cookies is 32 points. In her eyes, a fair share would be worth $10\,\dfrac{2}{3}$ points. The trimmed-down portion consists only of 8 vanilla cookies, which, in Rachel's eyes, is only worth 8 points. Since she views that portion as less than a fair share, she chooses not to trim it down any more.

STEP 3: The final step requires the last diminisher to get the piece. Since Julia was the last person to trim down the portion, she will receive the 8 vanilla cookies. To continue dividing the remaining cookies evenly, Rachel and Meredith can use the divide-and-choose method for two players. Note that this is only one possible solution to the problem. ■

Exercises

For Exercises 1 and 2, determine whether the fair-division problem is continuous, discrete, or mixed.

1. (a) A chocolate cake
   (b) A collection of antique place settings
   (c) An inheritance of a car and an acre of land

2. (a) A family dog
   (b) A collection of Picasso paintings
   (c) An inheritance of diamond jewelry, a vacation home, and an antique car

For Exercises 3 and 4, use the divide-and-choose method for two players.

3. Margie and George are sharing a chocolate bar that is half milk chocolate and half white chocolate. Margie likes both kinds of chocolate equally but George prefers white chocolate over milk chocolate. Margie has come up with three ways to divide the chocolate bar which are depicted in the following table:

	Portion 1	Portion 2
Option 1	100% milk chocolate	100% white chocolate
Option 2	50% milk chocolate & 50% white chocolate	50% milk chocolate & 50% white chocolate
Option 3	75% milk chocolate & 25% white chocolate	25% milk chocolate & 75% white chocolate

   (a) Under option 1, which portion would George choose and why?
   (b) Under option 2, which portion would George choose and why?
   (c) Under option 3, which portion would George choose and why?
   (d) Under option 1, suppose that George choose portion 1. How would both George and Margie judge the fairness of the portion they received compared with the value of the entire chocolate bar?

4. Jake and Owen are sharing a $\$6$ pizza for lunch. The pizza is half pepperoni and half cheese. Jake cooks the pizza and cuts it into 6 pieces, 3 pepperoni and 3 cheese. Jake likes pepperoni and cheese equally well but Owen prefers cheese to pepperoni 3 to 1. Jake has come up with two ways to divide the pizza which are depicted in the following table:

	Portion 1	Portion 2
Option 1	3 slices pepperoni	3 slices cheese
Option 2	2 slices pepperoni & 1 slice cheese	1 slice pepperoni & 2 slices cheese

   (a) Under option 1, which portion would Owen choose and why?

(b) Under option 2, which portion would Owen choose and why?

(c) Under option 1, suppose Owen chooses portion 2 and Jake receives portion 1. How would the two boys judge the fairness of the portion they received compared with the value of the entire pizza?

(d) Under option 2, suppose Owen chooses portion 2 and Jake receives portion 1. How would the two boys judge the fairness of the portion they received compared with the value of the entire pizza?

For Exercises 5 and 6, suppose you purchase a fruit plate at the grocery store for \$16. It is made up of 20 ounces of pineapple and 20 ounces of watermelon as shown in Figure 8.1.

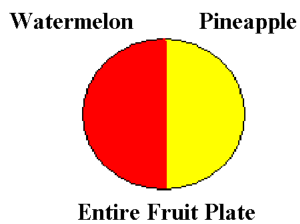


Figure 8.1: Exercises 5 and 6

5. Suppose you like watermelon three times as much as pineapple.

   (a) Determine your watermelon to pineapple preference ratio.

   (b) What fraction of the fruit plate is watermelon?

   (c) Based on your preference ratio, what fraction of the value of the fruit plate is the value of the watermelon?

   (d) Suppose you removed a portion of 14 ounces of watermelon. What dollar value would you place on that portion?

   (e) Refer to Figure 8.2. What dollar value would you place on portions II and III?

6. Suppose you like watermelon 5 times as much as pineapple.

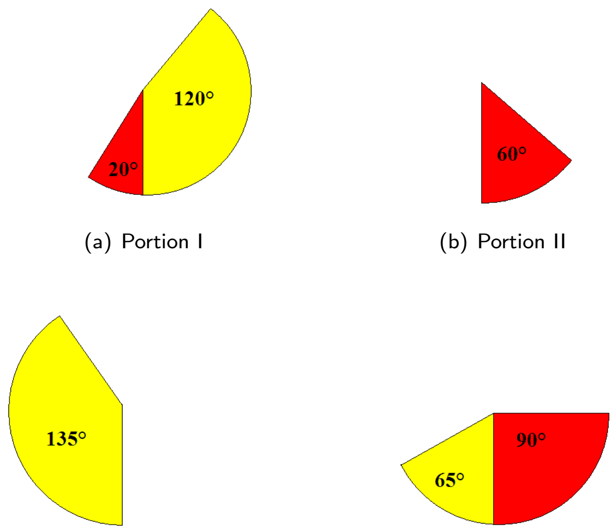
   (a) Determine your watermelon to pineapple preference ratio.

   (b) What fraction of the fruit plate is pineapple?

   (c) Based on your preference ratio, what fraction of the value of the fruit plate is the value of the pineapple?

   (d) Suppose you removed a portion of 16 ounces of pineapple. What dollar value would you place on that portion?

   (e) Refer to Figure 8.2. What dollar value would you place on portion I and IV?



(a) Portion I (b) Portion II

(c) Portion III (d) Portion IV

Figure 8.2: Exercises 5 and 6

7. An 8-pack of dinner rolls for $\$4.00$ contains 4 wheat rolls and 4 white rolls.

   (a) Sally likes both types of rolls equally well. Determine her white roll to wheat roll preference ratio. What dollar value would she place on each set of 4 rolls?

   (b) Randy prefers wheat rolls to white rolls 2 to 1. Determine his white roll to wheat roll preference ratio. What dollar value would he place on each set of 4 rolls?

   (c) Jenni only eats wheat rolls. Determine her white roll to wheat roll preference ratio. What dollar value would she place on each set of 4 rolls?

   (d) Sean prefers white rolls to wheat rolls 6 to 1. Determine his white roll to wheat roll preference ratio. What dollar value would he place on each set of 4 rolls?

8. Dove Manufacturing packaged a 16-ounce bottle of fragrance-free body wash with a 16-ounce bottle of scented body wash and priced it at $\$5.60$.

   (a) Darcy is allergic to fragrance and, therefore, never uses scented body wash but always uses fragrance-free body wash. Determine Darcy's fragrance-free to scented preference ratio. What dollar value would she place on each item?

(b) Erica will use either fragrance-free or scented body wash and, therefore, likes both equally well. Determine Erica's fragrance-free to scented preference ratio. What dollar value would she place on each item?

(c) Brandon prefers scented body wash over fragrance-free body wash 3 to 1. Describe what Brandon's preference ratio means. What dollar value would he place on each item?

(d) Josh prefers fragrance-free body wash to scented body wash 6 to 1. Describe what Josh's preference ratio means. What dollar value would he place on each item?

9. You buy a chocolate chip cookie at the bakery that has been dipped halfway into chocolate. You prefer the plain chocolate chip cookie over the chocolate dipped half 5 to 1. According to your own preferences, which of the divisions shown in the following table are a fair division?

	Portion 1	Portion 2
Division 1	$180°$ plain	$180°$ dipped
Division 2	$120°$ plain & $60°$ dipped	$60°$ plain & $120°$ dipped
Division 3	$150°$ plain & $30°$ dipped	$30°$ plain & $150°$ dipped
Division 4	$115°$ plain & $65°$ dipped	$65°$ plain & $115°$ dipped

For Exercises 10-17, suppose a bakery frosted a 9×9 cake with vanilla frosting on one side and chocolate frosting on the other side. Eric likes chocolate and vanilla frosting equally well, Alex prefers chocolate to vanilla frosting 3 to 1, Bailey will only eat vanilla frosting, and Rina will only eat chocolate frosting. Consider the following cake and the four portions shown in Figures 8.3 and 8.4.

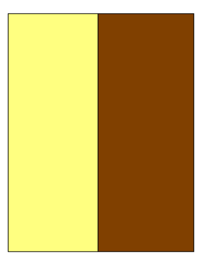
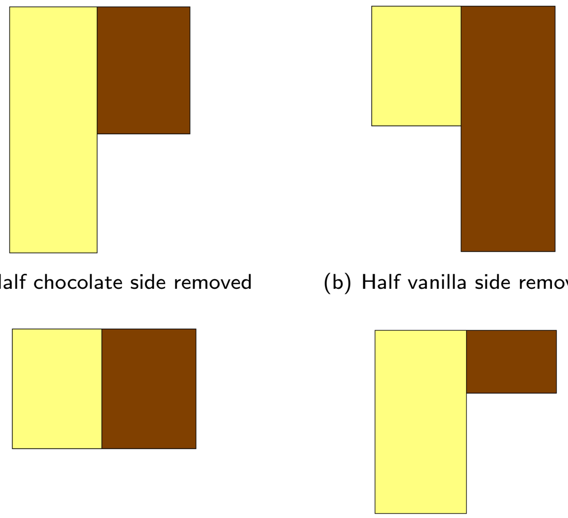


Figure 8.3: Entire Cake

10. Suppose that Eric bought the cake for $\$10.00$.

(a) What monetary value does he place on the chocolate half of the cake?



(a) Half chocolate side removed

(b) Half vanilla side removed

(c) Half of chocolate and vanilla sides removed

(d) Three-fourths of the chocolate side and one-fourth of vanilla side removed

Figure 8.4: Exercises 10-17

(b) What monetary value does he place on the vanilla half of the cake?

(c) What monetary value does he place on portions in Figure 8.4?

11. Suppose that Alex bought the cake for $\$10.00$.

    (a) What monetary value does he place on the chocolate half of the cake?

    (b) What monetary value does he place on the vanilla half of the cake?

    (c) What monetary value does he place on portions in Figure 8.4?

12. Suppose that Bailey bought the cake for $\$10.00$.

    (a) What monetary value does she place on the chocolate half of the cake?

    (b) What monetary value does she place on the vanilla half of the cake?

    (c) What monetary value does she place on portions in Figure 8.4?

13. Suppose that Rina bought the cake for $\$10.00$.

    (a) What monetary value does she place on the chocolate half of the cake?

    (b) What monetary value does she place on the vanilla half of the cake?

    (c) What monetary value does she place on portions in Figure 8.4?

14. Consider your answers from problem 10. For parts (a) and (b) below, describe a new portion of the cake containing vanilla and chocolate frosting to which Eric would assign the following dollar amounts.
    (a) $\$4.00$
    (b) $\$6.50$
15. Consider your answers from problem 11. For each parts (a) and (b) below, describe a new portion of the cake containing vanilla and chocolate frosting to which Alex would assign the following dollar amounts.
    (a) $\$4.00$
    (b) $\$6.50$
16. Consider your answers from problem 12. For each parts (a) and (b) below, describe a new portion of the cake containing vanilla and chocolate frosting to which Bailey would assign the following dollar amounts.
    (a) $\$4.00$
    (b) $\$6.50$
17. Consider your answers from problem 13. For each parts (a) and (b) below, describe a new portion of the cake containing vanilla and chocolate frosting to which Rina would assign the following dollar amounts.
    (a) $\$4.00$
    (b) $\$6.50$
18. Three kids, Mike, Matt, and Mary, are required to share the family car on the weekends. The car is available Friday nights, Saturday morning, Saturday nights, and Sunday afternoon. Mary comes up with three shifts, Shift A, Shift B, and Shift C, that she believes are a fair division of the weekend car time.
    (a) Suppose that Mike views Shift A as acceptable and Matt views Shifts B and C as acceptable. Describe a fair division of the time among Mike, Matt, and Mary.
    (b) Suppose that Mike and Matt view Shifts B and C as acceptable. Describe a fair division of the time among Mike, Matt, and Mary.
    (c) Suppose that Mike and Matt only view Shift C as acceptable. Describe how the three kids could create a fair division of the time amongst each other.
19. Jaime, Janice, and Joey are working to find a way to fairly divide 36 ounces of ice cream consisting of equal amounts of vanilla, chocolate, and strawberry. Jaime likes all three flavors equally well. Janice's preference ratio of vanilla to chocolate to strawberry is 1 to 2 to 3 and Joey's preference ratio is 1 to 2 to 1. Use the last-diminisher method and assume that Jaime serves the first portion and Janice takes the first turn in judging the fairness of that portion. How would the ice cream be divided fairly?

8.2 Discrete and Mixed Fair Division

The methods discussed in the previous section pertain to *continuous* fair-division problems. In this section, we will look at *discrete* fair-division problems, in which the item(s) being shared cannot be subdivided, and *mixed* fair-division problems, in which some items may be distributed and some may be subdivided.

Discrete Fair Division: The Method of Sealed Bids

Consider a situation in which two siblings, who live on opposite ends of the U.S., have inherited a car from an older family member. Obviously, it is not possible for the two siblings to share the car so one will inherit the car and will pay the other an amount of cash that is worth half the value of the car. How do they decide which sibling receives the car and how do they determine the car's value? It is necessary to determine the value and required cash payment before they decide who is receiving the car. Assuming that both siblings want the car and can afford to make the required cash payment, one fair way to determine the value and who receives the car is by using *the method of sealed bids* as illustrated in the next example.

Example 8.5. Bob and Sue have inherited their grandfather's car and they must determine who receives the car and it's value. Bob thinks to himself that the car is worth $\$2,200$ and Sue thinks to herself that the car is worth $\$2,900$. What should they do?

Solution. One reasonable way to determine the value of the car is to calculate the average of their estimates, or

$$\frac{\$2,200 + \$2,900}{2} = \$2,550$$

Thus, they have determined that the value of the car is $\$2,550$. Next, they need to decide who receives the car and who gets the cash.

Sue thought the car to be worth $\$2,900$, which is more than the $\$2,550$ value they agreed upon. Thus, Sue could buy out Bob's half-ownership for

$$\frac{\$2,550}{2} = \$1,275$$

which, to her, is a good deal. Bob would also be happy with the arrangement since he thought half the car to be worth

$$\frac{\$2,200}{2} = \$1,100$$

and he would be receiving $\$1,275$ from Sue in cash. Therefore, both siblings would be pleased if Sue received ownership of the car and Bob received the cash. ■

In Example 8.5, if Bob were to buy out Sue's half of the ownership, he would pay her $\$1,275$ which, to him, would seem too much. Sue, estimating the car's value at $\$2,900$, would also not be happy with the deal because she would not believe $\$1,275$ to be enough. Thus both would be unhappy if Bob bought out Sue.

Both players have incentive to make a fair bid because their estimated worth of the car affects the amount of cash that they are either paying or receiving. The main idea is to have all players make independent and confidential estimates on the value of the item(s), which we will call **bids**. A **sealed bid** allows each player to make his or her bid confidentially and place it in a sealed envelope. All envelopes are then opened at the same time so that no player knows the others' bids before making their own.

The player who is the highest bidder receives the item and buys out the others. The players with the lower bids do not receive the item but they receive a cash amount from the highest bidder. The steps of the Method of Sealed Bids are illustrated below.

The Method of Sealed Bids: Any number of players, n, are sharing any number of items, with the possibility of monetary compensation to guarantee fairness:

> **STEP 1:** All players confidentially and independently submit their bids stating a monetary value for each item – they make a sealed bid.
>
> **STEP 2:** The highest bidder on each item receives the item and that player pays the dollar amount of his or her bid to a **compensation fund**.
>
> **STEP 3:** Each player then receives $\frac{1}{n}$ of his or her bid on each item from the compensation fund.
>
> **STEP 4:** If there is money left over in the compensation fund, it is distributed evenly among all players.

Notice that there is no limit on the number of items on which one player makes the highest bid. Therefore, it is possible that one player receives most or all of the items and then must compensate all other players. In order for the method of sealed bids to be used, the players need to have enough cash to cover the possibility of compensating the other players. The next example will illustrate how the method of sealed bids can be used to divide multiple items among more than two players.

Example 8.6. Three brothers, Adam, Mike, and Paul, inherit a family home and a lake home. Their confidential and independent bids on each home appear in the following table. Using the method of sealed bids, determine who receives each property and the required amount of cash to fairly compensate the brothers who do not receive a property.

	Adam's Bid	Mike's Bid	Paul's Bid
Family home	$\$278,000$	$\$301,000$	$\$292,000$
Lake home	$\$203,000$	$\$181,000$	$\$196,000$

Solution. STEP 1: All brothers must submit their sealed bids of their private, honest estimates of the values of the properties. These bids appear in the above table.

STEP 2: Each property is assigned to the highest bidder. Therefore, Mike receives the family home for $\$301,000$ and he contributes that amount to the compensation fund. Adam receives the lake home for $\$203,000$ and he contributes that amount to the compensation fund. Now the compensation fund contains

$$\$301,000 + \$203,000 = \$504,000$$

STEP 3: The compensation fund is now distributed among the three brothers. Each brother receives one-third of the total of his bids from the compensation fund as shown:

$$\begin{array}{lcll} \text{Adam receives} & \dfrac{\$278,000+\$203,000}{3} & \approx & \text{\$160,333} \\ \text{Mike receives} & \dfrac{\$301,000+\$181,000}{3} & \approx & \text{\$160,666} \\ \text{Paul receives} & \dfrac{292,000+\$196,000}{3} & \approx & \text{\$162,666} \end{array}$$

Notice that the three brothers do not receive one-third of the value of the compensation fund.

STEP 4: The money leftover in the compensation fund is now distributed among the three brothers. The compensation fund now has

$$\$504,000 - \$160,333 - \$160,666 - \$162,666 = \$20,335$$

left. Each brother receives

$$\frac{\$20,225}{3} \approx \$6,778.33$$

from the compensation fund and the division is complete.

Adam receives the lake home for $\$203,000$ but he receives

$$\$160,333 + \$6,778.33 = \$167,111.33$$

from the compensation fund, so he pays a net amount of

$$\$203,000 - \$167,111.33 = \$35,888.67$$

for the lake home.

Mike receives the family home for $\$301,000$ but he receives

$$\$160,666 + \$6,778.33 = \$167,444.33$$

from the compensation fund, so he pays a net amount of

$$\$301,000 - \$167,444.33 = \$133,555.67$$

for the family home.

Lastly, Paul does not receive any property but he receives

$$\$162,666 + \$6,778.33 = \$169,444.33$$

in cash. ■

Although you can see that the brothers' dollar values of their inheritance differed, the division of property is proportional because each brother received one-third of what he considered to be the total value of the properties.

Discrete Fair Division: The Method of Points

Sometimes assigning a dollar amount to measure value or to compensate for those who do not receive a wanted item is inappropriate. Consider an example in which three couples are trying to decide who gets which of the three rooms at a very unique hotel. Normally, the couples would discuss which rooms offer certain desired features in order to assign rooms. It would not be appropriate to offer money for a desired room, nor would it be convenient to switch rooms in the middle of the day.

One way to use a fair-division procedure to solve a problem like this is to assign each player 100 points. (The number of points assigned to each player can differ by situation, but we will be using 100 points in our examples.) Each player then assigns a certain number of points to each alternative in order to specify their preferences and the intensity of those preferences. After players assign their points, the alternatives are distributed in a way to make all players happy, as measured by points. This fair-division procedure, the *method of points* is effective if the number of alternatives is the same as the number of players and every player is flexible.

Example 8.7. The mother of three boys, Jacob, Riley, and James, tells them that they must complete their chores before they can play with friends. She says that she needs the grass mowed, the bathroom cleaned, and the living room vacuumed, and she does not care which of the boys does which chore. The boys each assign points to each of the chores based on their preference, as shown in the following table. Use the method of points to determine which chore will be done by which boy.

	Jacob	Riley	James
Mow grass	32	60	29
Clean bathroom	48	18	35
Vacuum	20	22	36
Total	100	100	100

Solution. STEP 1: First the boys assign points to each chore based on their preferences. The results are shown in the above table.

STEP 2: Next we list all six possible arrangement and the corresponding points.

Possible Arrangement of Chores to the Brothers			Smallest Point Value
Jacob	Riley	James	
mow grass 32 points	clean bathroom 18 points	vacuum 36 points	18 points
mow grass 32 points	vacuum 22 points	clean bathroom 35 points	22 points
clean bathroom 48 points	mow grass 60 points	vacuum 36 points	36 points
clean bathroom 48 points	vacuum 22 points	mow grass 29 points	22 points
vacuum 20 points	mow grass 60 points	clean bathroom 35 points	20 points
vacuum 20 points	clean bathroom 18 points	mow grass 29 points	18 points

STEP 3: Now we consider the smallest-point-value column. In this case, exactly one arrangement yields a maximum smallest point value of 36 points. This arrangement assigning Jacob to clean the bathroom, Riley to mow the grass, and James to vacuum the living room, is the selected arrangement. All three boys received their first choice of chores and valued that chore to be at least one-third of the total number of points, thus making the division proportional. ■

We will later see that the method of points does not always guarantee a proportional division but it will get to as close to proportional as possible. The complete steps of the method of points are outlined on the next page.

The Method of Points: Three players are to determine who gets which of three items:

STEP 1: Each player assigns points to each item such that each player's points total 100.

STEP 2: List the six possible arrangements of players and items, including the point assignments for each item made by each player.

STEP 3: Note the smallest number of points assigned to an item by a player for each arrangement. If there is exactly one arrangement for which the smallest number is as large as possible, that is the winning arrangement. If there is more than one arrangement that has the maximum smallest number, a tie, continue on to step 4 with only those arrangements.

STEP 4: For the arrangements kept from step 3, note the second-to-largest number, the middle number, among the three that were assigned by the players. If there is one arrangement that has the maximum middle number, that is the winning arrangement. If there is more than one arrangement, another tie, continue on to step 5 with only those arrangements.

STEP 5: For the arrangement kept from step 4, note the largest number among the three that were assigned by the players. Select the arrangement for which the largest number is the maximum.

The method of points can be applied to more players as long as the number of players is still equal to the number of items being divided. However, as the number of players grows, the number of possible arrangements grows rapidly so we will limit our examples to including at most three players. As we have already seen, the method of points is effective for dividing desired items among multiple players. Consider the following Example.

Example 8.8. Couples A, B, and C are trying to determine which couple gets which room in the hotel. The following table shows the points that each couple assigned to which room when given 100 points. The higher the point value assigned to a room, the higher the preference of that room by that couple. Determine which couple gets which room by using the method of points.

	Couple A	Couple B	Couple C
Room 1	32	59	45
Room 2	61	17	29
Room 3	7	24	26
Total	100	100	100

Solution. STEP 1: The couples distribute their 100 points among the three alternatives as shown in the above table.

STEP 2: With three couples and three rooms, we have six possible assignments of rooms to couples. We will also note the smallest point value assignment made by any couple for any arrangement. These data are shown in the following table.

Possible Assignments of Couples to Rooms			Smallest Point Value
Couple A	Couple B	Couple C	
room 1 32 points	room 2 17 points	room 3 26 points	17 points
room 1 32 points	room 3 24 points	room 2 29 points	24 points
room 2 61 points	room 1 59 points	room 3 26 points	26 points
room 2 61 points	room 3 24 points	room 1 45 points	24 points
room 3 7 points	room 1 59 points	room 2 29 points	7 points
room 3 7 points	room 2 17 points	room 1 45 points	7 points

STEP 3: Now we will take a look at the smallest-point-value column which indicates the satisfaction of the couple who is the least happy with their room. First we will look for the largest number in this column which shows the arrangement that most pleases the least happy couple. The largest of these numbers is 26 which corresponds to the arrangement that assigns Couple A to room 2, Couple B to room 1, and Couple C to room 3.

Looking back at our first table, we notice that Couple A's first choice was room 2, Couple B's first choice was room 1, and Couple C's first choice was room 1. By selecting the arrangement with the largest number in the smallest-point-value column, we have granted two couples their first choice and one couple their third choice. Since both Couples B and C had room 1 for their first choice, although, based on points, Couple B felt more strongly about room 1. There was no possible arrangement that granted each couple their first choice; however, based on points and how strongly each couple felt about their first choice, this is the best arrangement. ■

Notice that the division in Example 8.8 is *not* proportional. Couple C valued room 3 at 26 points. For the division to be proportional, all couples should be assigned a room they value at 33 points or more.

Now let's take a look at an example in which ties exist and, therefore, extend the division process.

Example 8.9. Couples A, B, and C are to decide which couple gets which room in a unique hotel. The each couple is given 100 points to divide among the three rooms, as shown in the following table. Determine which couple gets which room by using the method of points.

	Couple A	Couple B	Couple C
Room 1	71	70	68
Room 2	12	10	12
Room 3	17	20	20
Total	100	100	100

Solution. STEP 1: First the three couples assign points to each of the rooms, as shown in the above table.

STEP 2: Next, we list all the possible arrangements of couples to rooms and the points assigned by each couple.

Possible Arrangements of Couples to Rooms			Smallest Point Value
Couple A	Couple B	Couple C	
room 1 71 points	room 2 10 points	room 3 20 points	10 points
room 1 71 points	room 3 20 points	room 2 12 points	12 points
room 2 12 points	room 1 70 points	room 3 20 points	12 points
room 2 12 points	room 3 20 points	room 1 68 points	12 points
room 3 17 points	room 1 70 points	room 2 12 points	12 points
room 3 17 points	room 2 10 points	room 1 68 points	10 points

STEP 3: Next we look at the smallest-point-value column and notice that there are four arrangements with the maximum smallest number, 12. Keeping these four arrangements only, we move on to step 4.

STEP 4: We now note the middle number in the four arrangements. These results are summarized in the following table.

Possible Arrangements of Couples to Rooms			Middle Point Value
Couple A	Couple B	Couple C	
room 1 71 points	room 3 20 points	room 2 12 points	20 points
room 2 12 points	room 1 70 points	room 3 20 points	20 points
room 2 12 points	room 3 20 points	room 1 68 points	20 points
room 3 17 points	room 1 70 points	room 2 12 points	17 points

We now have exactly three arrangements in which the largest middle number is 20. Keeping these three arrangements, we move on to step 5.

STEP 5: We now note the largest number in the three arrangement that we have kept. These results are summarized in the following table.

Possible Arrangements of Couples to Rooms			Largest Point Value
Couple A	Couple B	Couple C	
room 1 71 points	room 3 20 points	room 2 12 points	71 points
room 2 12 points	room 1 70 points	room 3 20 points	70 points
room 2 12 points	room 3 20 points	room 1 68 points	68 points

We now have exactly one arrangement that has the largest of the largest point values, 71. Thus, this is the arrangement that we should choose. Couple A should take room 1 (71 points), couple B should take room 3 (20 points), and couple C should take room 2 (12 points). Notice that this time, only one couple received their first choice since there were 2 ties. ■

Mixed Fair Division: The Adjusted-Winner Procedure

The last method we will discuss in this section allows some items to be assigned and some items to be divided – that is, a mixed fair division. It is a procedure that guarantees each player a fair share of the assets by, again, assigning points rather than monetary values.

This procedure, the *adjusted-winner procedure*, begins like the method of points by giving each player 100 points. The player who assigns the greatest number of points to a particular item achieves temporary ownership of that item. That player is then credited with the number of points that he or she assigned to that item. The main idea of this procedure is for the players to collect the same amount of points with a value equal to more than half of the total number of points. Then, the players have items of equal value based on their personal value systems creating a proportional fair division. The complete steps of the adjusted-winner procedure are outlined below.

The Adjusted-Winner Procedure for Two Players: Two players are to fairly divide any number of items. Items can be continuous or discrete. Ownership can be shared.

STEP 1: Each player assigns points to each item so that the total number of points for each player equals 100.

STEP 2: Each player temporarily receives ownership of the item(s) for which he or she assigned more points than the other player. The points corresponding to those items are added to the player's total. In the case that the two players assign the same number of points to a particular item, the item is given to the player who has the smallest point total based on the other items that have already been temporarily distributed.

STEP 3: If the two players X and Y do not have the same amount of points and, let's say, player X has a greater point total than player Y, select the item that is currently assigned to player X for which the ratio

$$\frac{\text{number of points assigned by player X}}{\text{number of points assigned by player Y}}$$

is the smallest. Then move ownership of that item from player X to player Y.

STEP 4: Reconsider the players' point totals and decide what to do next.

Case A: If the players' point totals are equal, we are done.

Case B: If player X still has more points than player Y, repeat step 3.

Case C: If player Y now has more points than player X, move a fraction of the item last moved from X to Y back to player X in order to achieve equality. The formula 8.1 is used in order to calculate what fraction of the item should be returned to player X.

The formula for Case C is given by

$$q = \frac{T_{\mathsf{Y}} - T_{\mathsf{X}} + P_{\mathsf{Y}}}{P_{\mathsf{X}} + P_{\mathsf{Y}}} \tag{8.1}$$

where

$$\begin{aligned}
q &= \text{fraction of the item to be moved from player Y back to player X}\\
T_{\mathsf{X}} &= \text{player X's point total (not including the item)}\\
T_{\mathsf{Y}} &= \text{player Y's point total (not including the item)}\\
P_{\mathsf{X}} &= \text{number of points assigned to the item by player X}\\
P_{\mathsf{Y}} &= \text{number of points assigned to the item by player Y}
\end{aligned}$$

The order in which items are moved is important and must be done correctly in order to maximize the number of points achieved by each player. Shifting the points may cause a different player to have too many points which is the reason for sharing the item as instructed in step 4. Note that in order to use the adjusted-winner procedure, all players must be willing to share ownership of an item.

Example 8.10. Players A and B have assigned a total of 100 points to a house, lake home, car, and boat that they are to divide fairly. Their distributed points are shown in the following table. Use the adjusted-winner procedure to fairly divide the items among players A and B.

	Player A	Player B
House	32	46
Lake home	23	6
Car	19	34
Boat	26	14
Total	100	100

Solution. STEP 1: First, the players assign points to each item as illustrated in the above table.

STEP 2: Next, each player receives the items to which he or she assigned the greatest amount of points and receives those points as well. Player A receives the lake home and the boat and player B receives the house and the car. This distribution and resulting points are shown in the following table.

	Player A		Player B
Lake home	23	House	46
Boat	26	Car	34
Total	49	Total	80

STEP 3: Since player B has more points than player A, we need to transfer an item from player B to player A. For each of player B's items, we need to calculate the ratio

$$\frac{\text{number of points assigned to item by player B}}{\text{number of points assigned to item by player A}}$$

as shown in the following table.

Ratios of Point Assignments	
House	$\frac{46}{32} \approx 1.44$
Car	$\frac{34}{19} \approx 1.79$

The calculated ratios illustrate the importance of the items to each player.

- If the ratio is equal to 1, it means that both players value the item the same.
- If the ratio is greater than 1, it means that player B values the item more than player A.
- If the ratio is less than 1, it means that player A values the item more than player B.

Since the ratio for the house is less than the ratio for the car, we will transfer the house ownership from player B to player A. The following table shows this new distribution.

	Player A		Player B
Lake home	23	Car	34
Boat	26		
House	32		
Total	81	Total	34

STEP 4: Now we need to reexamine the distribution. Player A now has more points than player B so we must use case C to determine the fraction of the house to be transferred back to player B. Some fraction, q, must be returned to player B. Thus, player A will receive $(1 - q)$ of the house. Our goal is to have players A and B end up with an equal number of points. We will use the formula 8.1. Thus, we have

$$\begin{aligned} q &= \text{fraction of the house that should return to player B} \\ T_{\mathsf{X}} &= \text{player B's point total, not including the house} \\ T_{\mathsf{Y}} &= \text{player A's point total, not including the house} \\ P_{\mathsf{X}} &= \text{number of points player B assigned to the house} \\ P_{\mathsf{Y}} &= \text{number of points player A assigned to the house} \end{aligned}$$

In our case, we have

$$T_{\mathsf{X}} = 34 \qquad T_{\mathsf{Y}} = 23 + 26 = 49 \qquad P_{\mathsf{X}} = 46 \qquad P_{\mathsf{Y}} = 32$$

We now substitute these values into the formula to calculate q.

$$\begin{aligned} q &= \frac{T_{\mathsf{y}} - T_{\mathsf{x}} + P_{\mathsf{y}}}{P_{\mathsf{x}} + P_{\mathsf{y}}} \\ &= \frac{49 - 34 + 32}{46 + 32} \\ &= \frac{47}{78} \\ &\approx 0.603 \end{aligned}$$

Thus, we know that 60.3% of the value of the house should be returned to player B. Therefore, player A will keep $100\% - 60.3\% = 39.7\%$ of the value of the house. This will make the point values for each player equal, which we will verify.

Player B now has

$$\begin{aligned} \text{Car points} + 60.3\% \text{ of house points} &= 34 + 0.603(46) \\ &\approx 61.74 \end{aligned}$$

Player A now has

$$\begin{aligned} &\text{Boat points} + \text{Lake home points} \\ &\qquad +39.7\% \text{ of house points} = 26 + 23 + 0.397(32) \\ &\qquad \approx 61.71 \end{aligned}$$

Due to previous rounding, the numbers may seem to be a little off; however, player A and player B now have an equal amount of total points, based on their own value systems, thus making the division proportional. Therefore, player A will receive the boat, the lake home, and 39.7% interest in the house, while player B should receive the car and 60.3% interest in the house. ■

Exercises

1. In the method of sealed bids, it is necessary that all players' bids are made without anyone knowing another person's bid. Explain why this is important.

2. It the method of sealed bids, at the time that all players are making their bids, no one knows whether or not they will receive the item they are bidding on. Explain how this encourages all players to make fair bids.

For Exercises 3-8, Josh and David inherit their grandfather's watch that is worth $\$1,000$. *Since it is not possible for both men to keep the watch, they use the method of sealed bids to determine who gets to keep it.*

3. Josh makes a bid of $\$1,200$ while David makes a high bid of $\$2,100$, even though he knows that the watch is worth much less. Explain why David has put himself at a huge disadvantage.

4. Josh makes a bid of $\$1,200$ while David makes an extremely low bid of $\$40$, even though he knows that the watch is worth much more. Explain why David has put himself at a huge disadvantage.

5. Josh makes a bid of $\$900$ while David makes a bid of $\$1,100$. Apply the method of sealed bids to determine the result.

6. Josh makes a bid of $\$1,200$ while David makes a bid of $\$1,000$. Apply the method of sealed bids to determine the result.

7. Suppose that David decides he is not interested in achieving ownership of the watch. If he sees that Josh made a bid of $\$1,200$, what should David bid on the watch, under these circumstances?

8. Suppose that David decides that he really does want the watch. If he sees that Josh made a bid of $\$1,200$, what should David bid on the watch, under these circumstances?

9. Two siblings, Mike and Mark, receive a computer from their parents. Each of the boys want to keep the computer in their own room. Mike values the computer at $\$800$ and Mark values the computer at $\$650$. They have decided to use the method of sealed bids to solve their problem.

   (a) Who should get to keep the computer in their room?

   (b) What should they consider to be the value of the computer?

   (c) What does Mike end up with, and why is he pleased with the results?

   (d) What does Mark end up with, and why is he pleased with the results?

10. Sarah and Shelley have inherited their mother's diamond necklace. Each of the women want to keep the necklace. Sarah values the necklace at $\$4,200$ and Shelley values the necklace at $\$3,900$. They have decided to solve the dispute using the method of sealed bids.

    (a) Who should get to keep the diamond necklace?

    (b) What should they consider the diamond necklace to be worth?

    (c) What does Sarah end up with, and why is she pleased with the results?

    (d) What does Shelley end up with, and why is she pleased with the results?

11. Tom, Jeff, and Roger inherit the family farm. They wish to keep the land in the family, rather than sell it and split the money. Independently, they each write their bids down and place them in a sealed envelope. Tom's bid is $\$800,000$, Jeff's bid is $\$860,000$, and Roger's bid is $\$680,000$. Use the method of sealed bids to assign the farm and monetary compensation.

12. Julia, Madeline, and Olivia inherit an antique bedroom set. They would like to keep it and continue passing it down through the family, rather than sell it and split the money. Independently, they each write their bids down and place

them in a sealed envelope. Julia's bid is $\$12,000$, Madeline's bid is $\$10,800$, and Olivia's bid is $\$12,400$. Use the method of sealed bids to assign the bedroom set and monetary compensation.

13. A group of three couples have decided to go on a vacation together. Once they reach their destination, they need to decided which couple gets which of the three available rooms. They have decided to use the method of points to solve their problem by allowing each couple to divide 100 points among the three available rooms, based on their preferences. The point assignments appear in the following table.

	Couple A	Couple B	Couple C
Room 1	26	18	19
Room 2	20	26	54
Room 3	54	56	27

(a) List the six possible arrangements of couples to rooms, and identify the smallest point value assigned to a room in each case.

(b) How many arrangements are there for which the smallest number is as large as possible?

(c) How should the rooms be assigned to the couples?

14. Three siblings, Alyssa, Julie, and Ron, have inherited three items from their great aunt's estate. They have decided to use the method of points to solve their problem by allowing each person to divide 100 points among the three items, based on their preferences. The point assignments appear in the following table.

	Alyssa	Julie	Ron
Piano	37	20	20
Painting	51	48	30
Desk	12	32	50

(a) List the six possible arrangement of sibling to item, and identify the smallest point value assigned to an item in each case.

(b) How many arrangements are there for which the smallest number is as large as possible?

(c) How should the items be assigned to the siblings?

15. Four boys were told that they are not allowed to play with their friends until they get their chores done. Their mother told them that they need to vacuum the living room, clean the bathroom, mow the lawn, and weed the garden. She has allowed them to decide who does which chore. The boys have decided to use the method of points to determine which chore is assigned to which boy. Their point assignments, each totaling 100 points for each boy, appear in the following table.

	Boy 1	Boy 2	Boy 3	Boy 4
Vacuum living room	19	24	10	32
Clean bathroom	25	35	30	34
Mow the lawn	24	31	35	24
Weed the garden	32	10	25	10

(a) Use the method of points to determine which chore is assigned to which boy.

(b) List each boy and determine whether they received their first, second, third, or fourth choice.

(c) After using the method of points, is the resulted division proportional? (As discussed in the first section.) Explain why or why not.

16. Four couples have shown up to Rent-A-Car on Saturday morning. The business only has four cars available: a black Mustang, a yellow Mustang, a red jeep, or a white truck. The point assignment appear in the following table.

	Couple 1	Couple 2	Couple 3	Couple 4
Black Mustang	9	9	29	9
Yellow Mustang	65	26	35	25
Red Jeep	16	40	10	40
White truck	10	25	26	26

(a) Use the method of points to determine which rental car is assigned to which couple.

(b) List each couple and determine whether they received their first, second, third, or fourth choice.

(c) After using the method of points, is the resulted division proportional? (As discussed in the first section.) Explain why or why not.

17. Kim and Joel are to divide the following items fairly: a desk, a painting, and a boat. They will use the adjusted-winner procedure. The following table shows the points that each has assigned to the items.

	Kim	Joel
Desk	45	55
Painting	15	10
Boat	40	35

(a) Based on the point assignments, who is tentatively given each item?

(b) Use the adjusted-winner procedure to determine the final results.

18. Terry and Rachel are to divide the following items fairly: a canoe, an armoire, and a sculpture. They will use the adjusted-winner procedure. The following table shows the points that each has assigned to the items.

	Terry	Rachel
Canoe	20	35
Armoire	55	35
Sculpture	25	30

(a) Based on the point assignments, who is tentatively given each item?

(b) Use the adjusted-winner procedure to determine the final results.

19. Wilson and Janet will use the adjusted-winner procedure to divide the following items fairly: a wool rug, a lamp, an antique painting, and a jewelry box. The following table shows their point assignments to each item.

	Wilson	Janet
Wool rug	34	38
Lamp	18	20
Painting	44	32
Jewelry Box	4	10

Complete the adjusted-winner procedure. Will Wilson and Janet have to share any of the items? How will the items be distributed fairly?

20. Paul and Rose will use the adjusted-winner procedure to divide the following items fairly: china, an antique table, a piano, a lake home, a boat, and a button collection. The following table shows their point assignments to each item.

	Paul	Rose
China	25	30
Table	20	9
Piano	20	23
Lake home	10	28
Boat	10	5
Button collection	15	5

Complete the adjusted-winner procedure. Will Paul and Rose have to share any of the items? How will the items be distributed fairly?

8.3 Envy-Free Division

In previous sections, we have discussed methods of fairly sharing assets or items among two or more players. We know that a player has received a fair share if, in a division among n players, every player has received a share worth at least $\frac{1}{n}$ of the total value of the assets or items. If every player has received a fair share, we say that the division is proportional. However, although a division may be proportional, it may still be the case that one player prefers the share given to another player over the share that they have received. In this section, we will discuss a method, *envy-free division*, that ensures a fair division and that all players are satisfied with their received portions.

Envy-free division is defined as a division in which each of the n players feels that he or she has received at least $\frac{1}{n}$ of the total value and that no other player has a share more valuable than his or her own.

In this section, we will introduce the envy-free division method for continuous fair-division problems involving three players. We have broken the method down into two parts – the first part distributes most of the "cake" and the second part distributes the excess that remains after part 1. The steps of the first part are illustrated below.

Continuous Envy-Free Division Method for Three Players, PART 1: To distribute the majority of the cake, Players A, B, and C are to divide and share a cake in an envy-free way.

STEP 1: Player A, the **divider**, divides the cake into three pieces that he or she considers to be of equal value.

STEP 2: Player B examines the three pieces and chooses the most valuable of the three pieces. (We will assume that player B only finds one piece to be the most valuable.)

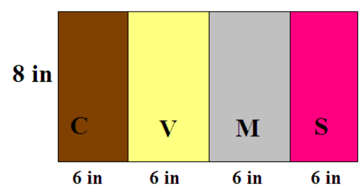
STEP 3: Player B, the **trimmer**, trims down the most valuable piece so that it is of equal value to the second-most-valuable piece. The part that was trimmed off the piece, the **excess**, is set aside for later.

STEP 4: Player C, the **chooser**, chooses the piece that he or she consider to be of the most value.

STEP 5: Player B then gets the piece that was trimmed if it is available. If not, player B gets any other piece that he or she considers to be of the most value.

STEP 6: Player A gets the remaining piece.

Example 8.11. Emily, Holly, Samantha want to share a cake that is one-fourth chocolate, one-fourth vanilla, one-fourth marble, and one-fourth strawberry, as shown below.



The following table shows their preference ratios for each of the cake flavors. Use part 1 of the continuous envy-free division method with Holly as the divider, Samantha as the trimmer, and Emily as the chooser.

	Emily	Holly	Samantha
Chocolate	1	1	1
Vanilla	3	1	3
Marble	4	2	1
Strawberry	2	2	1

Solution. We will assume that the players cut the cake by slicing it parallel to the lines separating the different flavors so that each piece will measure 8 inches long by some number of inches wide. Thus, the first piece includes chocolate and vanilla, the second piece includes vanilla and marble, and the third piece includes marble and strawberry.

STEP 1: First, Holly assigns point values to slices of the cake 8 inches long and 1 inch wide. Based on her preference ratio, she will assign 1 point for and 8 inch $\times 1$ inch slice of chocolate or vanilla cake, and she will assign 2 points for an 8 inch $\times 1$ inch slice of marble or strawberry cake. Each flavor of cake is 6 inches wide. Thus, Holly's point values for each of the flavored sections of the cake are shown in the following table.

Holly's Values

	Points per inch	Width in inches	Point Value
Chocolate	1	6	6
Vanilla	1	6	6
Marble	2	6	12
Strawberry	2	6	12
Total			36 points

In order for Holly to create three pieces that, to her, are of equal value, she must divide the cake in the following way:

Piece 1: 6 inches vanilla & 6 inches chocolate $= 12$ points

Piece 2: 6 inches marble $= 12$ points

Piece 3: 6 inches strawberry $= 12$ points

STEP 2: Next, Samantha, as the trimmer, examines the three pieces to determine which portion has the greatest value to her. Based on her preference ratio, she would assign 1 point for every 8 inch $\times$ 1 inch slice of chocolate, marble, or strawberry cake and she would assign 3 points for every 8 inch $\times$ 1 inch slice of vanilla cake. Thus, Samantha's point values for each of the pieces are shown in the following table.

Samantha's Values for Holly's Pieces

	Points per inch	Width in inches	Point Value
Piece 1			
Chocolate	1	6	6
Vanilla	3	6	18
Marble	1	0	0
Strawberry	1	0	0
Total			24 points
Piece 2			
Chocolate	1	0	0
Vanilla	3	0	0
Marble	1	6	6
Strawberry	1	0	0
Total			6 points
Piece 3			
Chocolate	1	0	0
Vanilla	3	0	0
Marble	1	0	0
Strawberry	1	6	6
Total			6 points

Based on Samantha's preferences, she finds that piece 1 is much more valuable than pieces 2 and 3.

STEP 3: Samantha must now trim down the first piece so that it is worth the same number of points as the second-largest piece. She trims down the first piece to include an 8 inches $\times\, 2$ inches slice of vanilla cake that is worth $3 \times 2 = 6$ points to her, the same point values of pieces 2 and 3. The trimmed off portion of piece 1, 8 inches $\times$ 4 inches of vanilla cake and 8 inches $\times$ 6 inches of chocolate cake, becomes "excess". Samantha has now created three new pieces.

Piece 1: 2 inches vanilla $= 6$ points

Piece 2: 6 inches marble $= 6$ points

Piece 3: 6 inches strawberry $= 6$ points

Excess: 4 inches vanilla & 6 inches chocolate $= 18$ points

STEP 4: The chooser, Emily, now assigns values to the pieces created by Samantha and chooses the piece that has the greatest amount of value to her. Based on her preferences, she would assign 1 point for every 8 inch $\times$ 1 inch slice of chocolate, 3 points for every 8 inch $\times$ 1 inch slice of vanilla, 4 points for every 8 inch $\times$ 1 inch slice of marble, and 2 points for every 8 inch $\times$ 1 inch slice of strawberry. Thus, Emily's point values for each of the pieces are shown in the following table.

Emily's Values for Holly's Pieces

	Points per inch	Width in inches	Point Value
Piece 1			
Chocolate	1	0	0
Vanilla	3	2	6
Marble	1	0	0
Strawberry	1	0	0
Total			6 points
Piece 2			
Chocolate	1	0	0
Vanilla	3	0	0
Marble	4	6	24
Strawberry	1	0	0
Total			24 points
Piece 3			
Chocolate	1	0	0
Vanilla	3	0	0
Marble	1	0	0
Strawberry	2	6	12
Total			12 points

Based on Emily's preferences, she finds that piece 2 is the most valuable to her and she chooses it.

STEP 5: The trimmer, Samantha, receives the trimmed piece, piece 1, since it is still available.

STEP 6: The divider, Holly, receives the last remaining piece, piece 3. ■

So, at the end of part 1, we have the following division:

Piece 1: 2 inches vanilla cake goes to Samantha

Piece 2: 6 inches marble cake goes to Emily

Piece 3: 6 inches strawberry cake goes to Holly

And the excess cake, put aside for part 2 of the division process, includes 4 inches of vanilla cake and 6 inches of chocolate cake. Part 2 of the continuous envy-free division method distributes the excess in a fair way that remains envy-free; the steps are outlined below.

Continuous Envy-Free Division Method for Three Players, PART 2: To share the excess, Players A, B, and C are to divide and share a cake in an envy-free way. Part 1 has already been completed with player A serving as the divider.

STEP 1: The player who received the trimmed piece, either player B or player C, will become the **second chooser**. The other becomes the **second divider**.

STEP 2: The second divider divides the excess into three pieces of equal value.

STEP 3: The second chooser chooses the piece that he or she considers to be of the most value.

STEP 4: Player A chooses the piece from the two remaining that he or she considers to have the greatest value.

STEP 5: The second divider then receives the last remaining piece of the excess.

Example 8.12. Emily, Holly, and Samantha would like to complete the envy-free division of the cake from Example 8.11. In part 1 of the division, Holly was the divider and Samantha received the trimmed piece. The excess that now needs to be divided include an 8 inch $\times$ 4 inch slice of vanilla cake and an 8 inch $\times$ 6 inch slice of chocolate cake. Apply part 2 of the continuous envy-free division method to divide the excess cake among Emily, Holly, and Samantha.

Solution. STEP 1: Since Samantha received the trimmed piece, she will be the second chooser. Emily will be the second divider.

STEP 2: Emily, as the second divider, must determine the total value of the excess cake and divide it into three equal pieces. Based on her preference, Emily finds the excess to be worth a total of 24 points, as shown in the following table.

Values assigned by Emily to excess cake

	Points per inch	Width in inches	Point Value
Vanilla	3	4	12
Chocolate	1	6	6
Total			18 points

To divide the excess cake into 3 fair pieces, Emily wants each piece to have a total value of 6 points. She divides the excess into the following three pieces, each equaling a total of 6 points:

Excess, Piece 1: 2 inches vanilla cake $= 6$ points

Excess, Piece 2: 2 inches vanilla cake $= 6$ points

Excess, Piece 3: 6 inches chocolate cake $= 6$ points

STEP 3: The second chooser, Samantha, chooses the piece that she considers to be the most valuable. Based on her preferences, her values assigned to each piece are shown below.

Values Samantha assigns to Emily's division of the excess

	Points per inch	Width in inches	Point value
Excess, Piece 1	3	2	6 points
Excess, Piece 2	3	2	6 points
Excess, Piece 3	1	6	6 points

Since all three pieces are of equal value to Samantha, arbitrarily, she chooses piece 3.

STEP 4: The original divider, Holly, chooses her piece next. Since the remaining two pieces, each consisting of an 8 inch $\times\, 2$ inch piece of vanilla cake, are identical, Holly arbitrarily chooses piece 1.

STEP 5: The second divider, Emily, gets the last remaining piece, piece 2.

The final results using the continuous envy-free division method are shown below.

Emily: An 8 inch $\times\, 6$ inch slice of marble cake and an 8 inch $\times\, 2$ inch slice of vanilla cake $= 30$ points

Holly: An 8 inch $\times\, 6$ inch slice of strawberry cake and an 8 inch $\times\, 2$ inch slice of vanilla cake $= 14$ points

Samantha: An 8 inch $\times\, 2$ inch slice of vanilla cake and an 8 inch $\times\, 6$ inch slice of chocolate cake $= 12$ points

Thus, we have achieved an envy-free fair division. ■

Exercises

1. When using the divide-and-choose method for two players, will the divider always receive a portion that is worth one-half of the total value? Explain.
2. When using the divide-and-choose method for two players, will the chooser always receive a portion that is worth one-half of the total value? Explain.

3. Jessie and Alicia have 6 ounces of chocolate ice cream and 6 ounces of cookies-and-cream ice cream. They wish to share the two flavors for their afternoon snack. Jessie prefers cookies-and-cream to chocolate 2 to 1. So she puts 4 ounces of cookies-and-cream and 1 ounce of chocolate ice cream in one bowl. She then fills the second bowl with 2 ounces of cookies-and-cream and 5 ounces of chocolate ice cream in the second bowl. Since she split the ice cream into two bowls, Alicia gets to choose which bowl she wants. She prefers cookies-and-cream to chocolate 3 to 1.

   (a) What value will Jessie place on each of the bowls of ice cream?

   (b) What value will Alicia place on each of the bowls of ice cream?

   (c) Is the division proportional? Why?

   (d) Is the division envy-free? Explain.

4. Joseph and Jason have 7 vanilla cookies and 10 chocolate cookies to share for their afternoon snack. Joseph prefers vanilla to chocolate 3 to 1 and Jason prefers vanilla to chocolate 4 to 3. Joseph decides to put 5 vanilla cookies and $\frac{1}{2}$ a chocolate cookie on one plate and the remaining cookies on the second plate. Jason gets to choose which plate he wants.

   (a) What value would Joseph place on each plate of cookies?

   (b) What value would Jason place on each plate of cookies?

   (c) Is the division proportional? Why?

   (d) Is the division envy-free? Explain.

For Exercises 5-10, three hungry boys, Paul, Justin, and Richard, are sharing a half-pepperoni, half-cheese pizza. Paul prefers pepperoni to cheese 2 to 1, Justin prefers pepperoni to cheese 4 to 1, and Richard prefers pepperoni to cheese 3 to 2. Using the lone-divider method, also known as the divide-and-choose method for three players, Paul is the lone divider and splits the pizza into three servings: (1) 50° slice of pepperoni and 80° slice of cheese; (2) 90° slice of pepperoni; (3) 40° slice of pepperoni and 100° slice of cheese.

5. According to each boy, determine how many points represent a fair share.

6. Verify that Paul created three servings that he believes to be of equal value.

7. What value would Justin place on each of the three servings created by Paul? Which serving would be most valuable to him?

8. What value would Richard place on each of the three servings created by Paul? Which serving would be most valuable to him?

9. Describe how the servings would be distributed.

10. Explain why they have achieved a proportional fair division.

For Exercises 11-16, suppose the boys from Exercises 5-10 the previous situation are sharing the same pizza that is half-pepperoni, half-cheese. But this time, suppose Paul prefers pepperoni to cheese 2 to 1, Justin prefers pepperoni to cheese 1 to 4, and Richard prefers pepperoni to cheese 3 to 1. Suppose Paul is still the lone divider and splits the pizza into three servings: (1) $50°$ slice of pepperoni and $80°$ slice of cheese; (2) $90°$ slice of pepperoni; (3) $40°$ slice of pepperoni and $100°$ slice of cheese.

11. According to each boy, determine how many points represent a fair share.
12. Verify that Paul created three servings that he believes to be of equal value.
13. What value would Justin place on each of the three servings created by Paul? Which serving would be most valuable to him?
14. What value would Richard place on each of the three servings created by Paul? Which serving would be most valuable to him?
15. Describe how the servings would be distributed and determine whether the boys have achieved a proportional fair division. Explain.
16. Is the final division of servings envy-free? Why or why not?
17. Describe a situation in which the last-diminisher method leads to a fair division that is envy-free.
18. Describe a situation in which the last-diminisher method leads to a fair division that is proportional, but not envy-free.
19. A family of three members is to share three different desserts, cheesecake, ice cream, and apple pie. The family members and their preferences for the three types of desserts are given in the following table, where larger numbers signify a greater preference.

	Family member 1	Family member 2	Family member 3
Cheesecake	3	3	3
Ice cream	1	4	1
Apple Pie	5	2	2

    If family member 3 is the divider and Family member 1 is the chooser, find a proportional, envy-free division of the three desserts among the three family members.

20. You like hawaiian pizza three times as much as either pepperoni or cheese pizza. Three plates of pizza have been served:

    Plate 1: 1 slice pepperoni, 4 slices cheese, and 3 slices hawaiian

    Plate 2: 2 slices pepperoni, 4 slices cheese, and 1 slice hawaiian

    Plate 3: 3 slices pepperoni, 1 slice cheese, and 5 slices hawaiian

(a) According to your preferences, calculate the point value for each of the plates of pizza. Determine which is the plate of the greatest value and which is the plate of the second-greatest value.

(b) Identify two ways to trim the plate of the greatest value to match the value of the second-greatest valuable plate.

Answers to selected Exercises

1.1 Fractions, pages 4–5

1. $\frac{2}{3}$

6. 180

14. $\frac{2}{3}$

1.2 Percents, page 8

4. 0.0048

11. 135%

18. 512.9

24. $\$450$

27. $\$637.33$

1.3 Logarithms, pages 10–10

3. -1

7. 1.441

12. 0.290

1.4 Counting, pages 18–19

3. $10,000$

9. 1820

3.1 Probability Basics and Simple Experiments, pages 63–65

3. (a) 8
   (b) $-\frac{140}{3}$
   (c) $-\frac{35}{12}$
   (d) $\frac{82}{63}$
5. (a) HHH, HHT, HTH, HTT, THH, THT, TTH, TTT
   (b) $\frac{1}{8}$
   (c) $\frac{1}{8}$
   (d) $\frac{3}{8}$
10. $\frac{5}{36}$

3.2 Probability of Events Involving "Not" and "Or", pages 70–72

2. (a) $\frac{1}{4}$
   (b) $\frac{3}{4}$
   (c) $\frac{1}{2}$
   (d) $\frac{1}{2}$
6. 0.2
11. $\frac{8}{13}$
18. (a) odds against – $10:3$
   odds in favor – $3:10$
   (b) odds against – $3:1$
   odds in favor – $1:3$
   (c) odds against – $12:1$
   odds in favor – $1:12$
   (d) odds against – $10:3$
   odds in favor – $3:10$

20. (a) $\frac{5}{13}$

    (b) $\frac{2}{3}$

3.3 Conditional Probability and Events Involving "And", pages 76–78

7. $\frac{1}{6}$

12. The probability that the person is a senior given that the person is a woman.

17. (a) $\{(1, G), (2, P), (3, G), (4, P), \ldots, (59, G), (60, P)\}$

    (b)
    $$P(A) = \frac{1}{2}$$
    $$P(B) = \frac{1}{4}$$
    $$P(A \cup B) = \frac{3}{4}$$
    $$P(A \cap B) = 0$$

    (c) yes

    (d) $P(A) + P(B) = \frac{1}{2} + \frac{1}{4} = P(A \cup B)$

22. (a) $\frac{25}{169}$

    (b) $\frac{33}{221}$

3.4 Expected Value, pages 80–82

3. $-\$1.25$

7. $\$300$

4.1 Graphs, pages 87–90

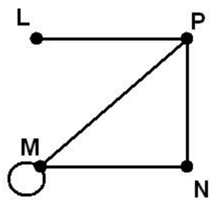
4. (a) $\mathcal{V} = \{G, H, S, T, V, K\}$

   (b) $\mathcal{E} = \{GG, GH, HV, HV, TV, VS, VK, SK\}$

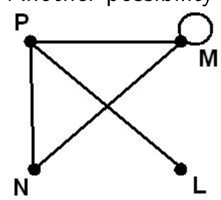
   (c)

Vertex	Degree
G	3
H	3
S	2
T	1
V	5
K	2

8. One possibility is



Another possibility is



4.2 Euler's Theorems, pages 93–94

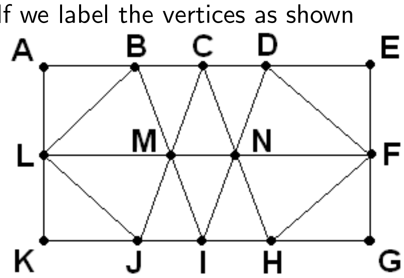
3. This graph has neither since the graph has four odd vertices.

7. This graph has an Euler path because it has two odd vertices.

10. This graph has neither because it is not connected.

4.3 Fleury's Algorithm, pages 97–98

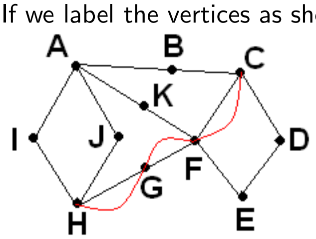
6. If we label the vertices as shown



Then one possible Euler path is $LABCMLKJINDEFNHGF$.

4.4 Eulerizing Graphs, pages 101–102

4. If we label the vertices as shown



We can add duplicate edges CF, FG, and GH. Then one possible eulerization is $AIHJABCDEFCFGHGFKA$.

4.5 Networks, pages 109–110

10. 36

5.1 Interest, pages 120–123

1. (a) $\$1,050$
   (b) $\$8,050$
7. $\$26.25$
14. (a) $\$12,480$
   (b) $\$480$
21. $\$10,117.80$
25. (a) $\$25,022.59$
   (b) $\$25,230.41$

   Compounding continuously gives a greater return.
28. 4.5%
35. 2010:$\$387,570.34$
   2020:$\$539,097.99$
   2030:$\$1,152,740.41$
   2040:$\$9,896,113.21$

5.2 Loans, pages 128–131

3. (a) $\$699$
   (b) $\$1,560$
   (c) $\$261$
11. (a) $\$101.55$
   (b) $\$678.55$
14. (a) $\$3.08$
   (b) $\$4.18$
   (c) $\$3.23$

5.3 Mortgages, pages 137–139

4. $\$63,720$
11. Maximum Home Price: $\$300,000$;
   Monthly Expenses: $\$2,083.33$
13. Maximum Home Price: $\$255,000$;
   Monthly Expenses: $\$1,770.83$
16. $\$9,340$

22. $\$2,394$

23. $\$1,170$

25. $\$59,000$

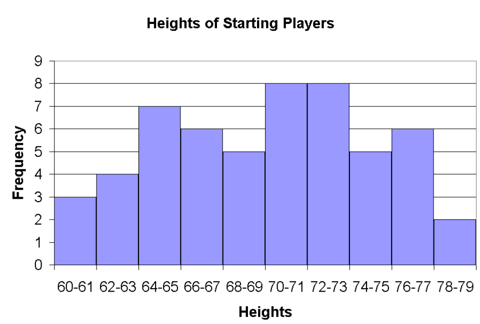
6.1 Population and Samples, pages 149–151

5. 70

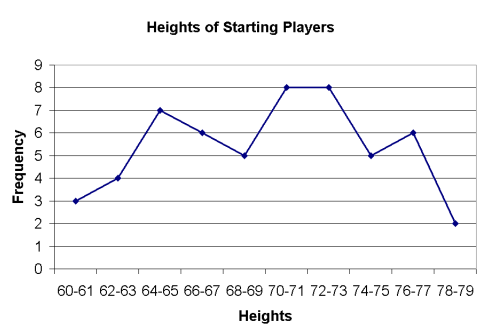
9. (a)

Height	f	$\frac{f}{n}$
$60-61$	3	0.056
$62-63$	4	0.074
$64-65$	7	0.130
$66-67$	6	0.111
$68-69$	5	0.093
$70-71$	8	0.148
$72-73$	8	0.148
$74-75$	5	0.093
$76-77$	6	0.111
$78-79$	2	0.037
	$n=54$	

(b)



(c)



6.2 Measures of Central Tendency, pages 156–157

1. (a) 18.1
   (b) 15
   (c) 10

6.3 Measures of Dispersion, pages 159–160

1. 19

5. (a) 81
   (b)

Data item	Deviation
75	-6
85	4
80	-1
75	-6
90	9

   (c) 170

10. 2.24

6.4 Normal Distribution, page 169

1. 68%

8. $-\frac{1}{2}$

14. (a) 21.19
    (b) 78.81

6.5 Scatter Plots, Correlation, and Regression Lines, pages 174–175

5. (d)

11. $r \approx 0.86$, $y = 0.84x + 4.3$

7.1 Voting Systems, pages 183–188

1. (a) Peterson
   (b) Holen – 26.4%
   Rodney – 23.1%
   James – 10.7%
   Peterson – 39.7%
   (c) 61 votes

(d) no

3. (a) 8 votes
   (b) Holiday Inn – 5
   Park – 3
   Bowling Alley – 6
   (c) Bowling Alley

14. (a) Jim
    (b) Jim

16. 5 candidates – 10 comparisons
    8 candidates – 28 comparisons

17. 6 rankings:
    ABC, ACB, BAC
    BCA, CAB, CBA

7.2 Flaws in the Voting Systems, pages 195–199

1. (a) A
   (b) B
   (c) majority, head-to-head

4. C wins but B is preferred to C in a head-to-head comparison.

8. A over B $7 - 4$
   C over A $6 - 5$
   C over B $7 - 4$

   C wins

11. A

13. B

7.3 Weighted Voting Systems, pages 205–207

3. $[27|13, 9, 8, 7, 7, 4, 2, 2, 1]$

7. $[10|5, 5, 3, 2, 2, 2]$

9. (a) 4
   (b) 14
   (c) 28
   (d) 14

11. (a) winning
    (b) losing
    (c) winning

14. (a) winning
    (b) losing
    (c) winning

8.1 Divide and Choose Methods, pages 218–223

1. (a) continuous
   (b) discrete
   (c) mixed

3. (a) Portion 2
   (b) Either
   (c) Portion 2
   (d) Margie would view her portion as fair but George would view his portion as less than fair.

10. (a) $\$5$
    (b) $\$5$
    (c) Portion I – $\$7.50$
    Portion II – $\$7.50$
    Portion III – $\$5.00$
    Portion IV – $\$5.00$

13. (a) $\$10$
    (b) $\$0$
    (c) Portion I – $\$5.00$
    Portion II – $\$10.00$
    Portion III – $\$5.00$
    Portion IV – $\$2.50$

17. (a) 60% of chocolate removed
    (b) 35% of chocolate removed

8.2 Discrete and Mixed Fair Division, pages 236–240

5. David – watch

   Josh – $\$500$

11. Tom – $\$293,333.33$

    Jeff – farm

    Roger – $\$253,333.33$

14. (a)

Alyssa	Julie	Ron	Smallest Points
Piano	Painting	Desk	37
Piano	Desk	Painting	30
Painting	Piano	Desk	20
Painting	Desk	Piano	20
Desk	Piano	Painting	12
Desk	Painting	Piano	12

    (b) 1

    (c) Alyssa – piano
        Julie – painting
        Ron – desk

17. (a) Kim – painting, boat
        Joel – desk

    (b) Kim – painting, boat
        Joel – desk

8.3 Envy-Free Division, pages 246–249

6. (1) 180

   (2) 180

   (3) 180

8. (1) : 310

   (2) : 270

   (3) : 320

   Richard would value serving (3) the most.

13. (1) 370

    (2) 90

    (3) 440

    Most valuable – serving (3)

20. (a) Plate 1 – 14

Plate 2 – 9
Plate 3 – 19

(b) Possible options:

Take away 1 Hawiaiian piece and 2 pepperoni pieces
Take away 1 Hawiaiian piece, 1 cheese piece, and 1 pepperoni piece
Take away 4 cheese pieces and 1 pepperoni piece

Index