Existence of Positive Solutions of a Second Order Right Focal Boundary Value Problem

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Abstract

This paper deals with the existence of a positive solution of a second order right focal boundary value problem. In particular, this paper profiles an application of a recent generalization of the Leggett-Williams fixed point theorem, which is void of any invariance like conditions, by utilizing operators and functionals. The sets in the fixed point theorem are defined using operators by an ordering generated by a border symmetric set, which leads to functional type boundaries of positive solutions of the boundary value problem over some parts of the domain. An example is provided to demonstrate the advantages of the flexibility of the results and the fixed point theorems.
**Key words:** Fixed point theorems, Leggett-Williams, expansion, compression, operator-functional type, positive solution, right focal, boundary value problem.

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1 Introduction

Generalizations of the Leggett-Williams fixed point theorem [13] have received tremendous recent attention. Many results involve functionals and norms; see [1, 8, 6, 12, 16] for some of these generalizations. Mavridis [14] published the first extension of the Leggett-Williams fixed point theorem that used concave and convex operators, however the results incorporated invariance-like conditions. Anderson, Avery, Henderson and Liu [3] published the first operator type expansion-compression fixed point theorem void of invariance like conditions. To eliminate the comparability restriction in [3], and at the same time, to keep the functional boundaries, Anderson, Avery, Henderson and Liu [4] created a hybrid operator-functional fixed point theorem by introducing a less restrictive ordering through a border symmetric set. In this paper, we will apply this fixed point theorem from [4] utilizing functionals and operators to a second order right focal boundary value problem. The techniques demonstrated here show the flexibility of the theorem.

2 Preliminaries

In this section, some preliminary definitions and the fixed point theorem from [4] that will be used in our applications are stated.

**Definition 2.1** Let $E$ be a real Banach space. A nonempty, closed, convex set $P \subseteq E$ is called a cone if, for all $x \in P$ and $\lambda \geq 0$, $\lambda x \in P$, and if $x, -x \in P$ then $x = 0$.

Every subset $C$ of a Banach space $E$ induces an ordering in $E$ given by $x \leq_C y$ if and only if $y - x \in C$, and we say that $x <_C y$ whenever $x \leq_C y$ and $x \neq y$. Furthermore, if the interior of $C$, which we denote as $C^\circ$, is nonempty then we say that $x \ll_C y$ if and only if $y - x \in C^\circ$. Note that if $C$ and $D$ are subsets of a Banach space $E$ with $C \subseteq D$ then

$$x \leq_C y \text{ implies } x \leq_D y.$$ 

**Definition 2.2** A closed, convex subset $M$ of a Banach space $E$ with nonempty interior is said to be a border symmetric subset of $E$ if for all $x \in M$ and $\lambda \geq 0$, $\lambda x \in M$, and if the order induced by $M$ satisfies the property that $x \leq_M y$ and $y \leq_M x$ implies that $x - y \not\in M^\circ$ and $y - x \not\in M^\circ$.

The border symmetric property is a less restrictive replacement of the antisymmetric property of a partial order.

**Definition 2.3** An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.
Definition 2.4 Let $P$ be a cone in a real Banach space $E$. Then we say that $A : P \to P$ is a continuous concave operator on $P$ if $A : P \to P$ is continuous and

$$tA(x) + (1-t)A(y) \leq_P A(tx + (1-t)y)$$

for all $x, y \in P$ and $t \in [0, 1]$. Similarly we say that $B : P \to P$ is a continuous convex operator on $P$ if $B : P \to P$ is continuous and

$$B(tx + (1-t)y) \leq_P tB(x) + (1-t)B(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Definition 2.5 A map $\alpha$ is said to be a nonnegative continuous concave functional on a cone $P$ of a real Banach space $E$ if $\alpha : [0, \infty) \to P$ is continuous and

$$t\alpha(x) + (1-t)\alpha(y) \leq \alpha(tx + (1-t)y)$$

for all $x, y \in P$ and $t \in [0, 1]$. Similarly we say the map $\beta$ is a nonnegative continuous convex functional on a cone $P$ of a real Banach space $E$ if $\beta : [0, \infty) \to P$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Let $R$ and $S$ be operators on a cone $P$ of a real Banach space $E$, and let $Q$ and $M$ be subsets of $E$ that contain $P$, with $x_R, x_S \in E$. Then we define the sets,

$$P_Q(R, x_R) = \{ y \in P : R(y) \ll_Q x_R \}$$

and

$$P(R, S, x_R, x_S, Q, M) = P_Q(R, x_R) - P_M(S, x_S).$$

Definition 2.6 Suppose $P$ is a cone in a real Banach space $E$, $Q$ is a subset of $E$ with $P \subset Q$, $\alpha$ is a nonnegative continuous concave functional on $P$, $B$ is a continuous convex operator on $P$, $a$ is a nonnegative real number, $y_B \in E$ and $T : P \to P$ is a completely continuous operator. Then we say that $T$ is LW-inward with respect to $P_Q(B, \alpha, y_B, a)$, if $P_Q(B, y_B)$ is bounded and the following conditions are satisfied:

(B1) $\{ y \in P : a < \alpha(y) \text{ and } B(y) \ll_Q y_B \} \neq \emptyset$.

(B2) If $y \in \partial P_Q(B, y_B)$ and $a \leq \alpha(y)$, then $B(Ty) \ll_Q y_B$.

(B3) If $y \in \partial P_Q(B, y_B)$ and $\alpha(Ty) < a$, then $B(Ty) \ll_Q y_B$.

Definition 2.7 Suppose $P$ is a cone in a real Banach space $E$, $M$ is a border symmetric subset of $E$ with $P \subset M$, $\beta$ is a nonnegative continuous convex functional on $P$, $A$ is a continuous concave operator on $P$, $b$ is a nonnegative real number, $y_A \in E$ and $T : P \to P$ is a completely continuous operator. Then we say that $T$ is LW-outward with respect to $P_M(\beta, A, b, y_A)$, if $P_M(A, y_A)$ is bounded and the following conditions are satisfied:
\((A1)\) \(\{ y \in P : y_A \ll_M A(y) \text{ and } \beta(y) < b \} \neq \emptyset.\)

\((A2)\) If \(y \in \partial P_M(A, y_A)\) and \(\beta(y) \leq b\), then \(y_A \ll_M A(Ty)\).

\((A3)\) If \(y \in \partial P_M(A, y_A)\) and \(b < \beta(Ty)\), then \(y_A \ll_M A(Ty)\).

We will exhibit the existence of a positive solution of a right focal boundary value problem as a result of applying the fixed point theorem utilizing operators and functionals [4] of Anderson, Avery, Henderson and Liu which appears as the following theorem (Theorem 2.8).

**Theorem 2.8** Suppose \(P\) is a cone in a real Banach space \(E\), \(Q\) is a subset of \(E\) with \(P \subset Q\), \(M\) is a border symmetric subset of \(E\) with \(P \subset M\), \(\alpha\) is a nonnegative continuous concave functional on \(P\), \(\beta\) is a non-negative continuous convex functional on \(P\), \(B\) is a continuous convex operator on \(P\), \(A\) is a continuous concave operator on \(P\), \(a\) and \(b\) are nonnegative real numbers, and \(y_A\) and \(y_B\) are elements of \(E\). Furthermore, suppose that \(T : P \rightarrow P\) is completely continuous and

\((D1)\) \(T\) is LW-inward with respect to \(P_Q(B, \alpha, y_B, a)\):

\((D2)\) \(T\) is LW-outward with respect to \(P_M(\beta, A, b, y_A)\).

If

\((H1)\) \(P_M(A, y_A) \subsetneq P_Q(B, y_B)\), then \(T\) has a fixed point \(y \in P(B, A, y_B, y_A, Q, M)\), whereas, if

\((H2)\) \(P_Q(B, y_B) \subsetneq P_M(A, y_A)\), then \(T\) has a fixed point \(y \in P(A, B, y_A, y_B, M, Q)\).

### 3 A Second Order Right Focal Boundary Value Problem

In this section, we make application of Theorem 2.8 to obtain a positive solution of the second order right focal boundary value problem,

\[
\begin{align*}
x'' + f(x) &= 0, \quad t \in (0, 1), \\
x(0) &= x'(1) = 0,
\end{align*}
\]

where \(f : \mathbb{R} \rightarrow [0, \infty)\) is continuous. Right focal boundary value problems have received a lot of attention, see [2, 9, 5, 7, 10, 11, 15, 17] for some representable results. Our results take a step towards global behaviors of the solution (knowledge about the solution on the domain) instead of local behaviors of the solution (knowledge about the solution at a collection of points).

The Green’s function for \(-x'' = 0\) satisfying (3.2), is given by

\[G(t, s) = \min\{t, s\}, \quad (t, s) \in [0, 1] \times [0, 1].\]

Notice that, for any \(s \in [0, 1]\), \(wG(y, s) \geq yG(w, s)\) if \(0 \leq y \leq w \leq 1\), and \(G(t, s)\) is nondecreasing in \(t\) for each fixed \(s \in [0, 1]\). It is well known that if \(x\) is a fixed point of the operator \(T\) defined by

\[Tx(t) = \int_0^t G(t, s)f(x(s))ds,\]
then $x$ is a solution of the BVP (3.1), (3.2).

Define the Banach space $E = C[0, 1]$ with the supremum norm, and let the cone $P \subset E$ be defined by

$$P := \{ x \in E \mid x \text{ is nonnegative, nondecreasing, and concave} \}.$$ 

If $x \in P$, then for all $0 \leq t_1 \leq t_2 \leq 1$,

$$t_2 x(t_1) \geq t_1 x(t_2). \quad (3.3)$$

By the properties of the Green’s function $G(t, s)$, we have $(Tx''(t) = -f(x(t))) \leq 0$, $(Tx)(t) \geq 0$, and $(Tx)(0) = (Tx)(1) = 0$, which imply $T : P \to P$. Also, by an Arzela-Ascoli Theorem argument, it is straightforward that $T$ is a completely continuous operator.

**Theorem 3.1** Suppose there exist $\tau \in (0, 1]$, $0 < \mu_1 \leq \mu_2 \leq 1$, $y_A, y_B \in P \setminus \{0\}$, $0 < a < y_B(1) \tau$, $\frac{y_A(\mu_1)}{\mu_1} \mu_2 < b$ and $f : [0, \infty) \to [0, \infty)$ continuous, such that $f$ satisfies at least one of (i) or (i'), where

(i) $f$ is nondecreasing over $[a, y_B(1)]$ and $f(w) < \min_{t \in [\tau, 1]} \frac{y_B(t) - t f(y_B(s)) ds}{\tau^2}$ for $w \in [0, y_B(\tau)]$,

(ii) $f$ is nonincreasing over $[0, y_B(\tau)]$ and $f(w) < \min_{t \in [\tau, 1]} \frac{y_B(t) - t f(y_B(s)) ds}{t (1 - \tau)}$ for $w \in [a, y_B(1)]$,

and $f$ satisfies at least one of (ii) or (ii'), where

(ii') $f(w) > \max_{t \in [\mu_1, \mu_2]} \frac{y_A(t)}{t (1 - \mu_2)}$ for $w \in [y_A(\mu_1), y_A(\mu_2)]$.

(ii') $f(w) > \frac{6 y_B^2}{(\mu_2 - \mu_1)^2 t_1^2} w$ for $w \in [0, y_A(\mu_2)]$.

If $y_A(\mu_1) < y_B(1) \min(\tau, \mu_1)$, then (3.1), (3.2) has a solution $y^* \in P$ such that $y^*(t) < y_B(t)$, for $t \in [\tau, 1]$, and $y^*(t) \not< y_A(t)$, for $t \in [\mu_1, \mu_2]$.

If $y_B(1) < y_A(\mu_1)$, then (3.1), (3.2) has a solution $y^{**} \in P$ such that $y^{**}(t) < y_A(t)$, for $t \in [\mu_1, \mu_2]$, and $y^{**}(t) \not< y_B(t)$, for $t \in [\tau, 1]$.

**Proof:** We define $A$ and $B$ both be the identity operator on $P$. Then, $A$ and $B$ are continuous linear operators mapping $P$ to $P$ and also are concave and convex, respectively. Let the sets $M$ and $Q$ be defined by

$$M = \{ y \in E \mid y(t) \geq 0, \ t \in [\mu_1, \mu_2] \} \quad \text{and} \quad Q = \{ y \in E \mid y(t) \geq 0, \ t \in [\tau, 1] \}.$$ 

It is easy to see that $M$ is a border symmetric subset of $E$ since if $x, -x \in M$ then for all $t \in [\mu_1, \mu_2]$ we have that $x(t) \geq 0$ and $-x(t) \geq 0$ hence $x(t) = 0$ for all $t \in [\mu_1, \mu_2]$ hence $x \in \partial M$. Also note that $P \subset M$, $P \subset Q$ with

$$M^\circ = \{ y \in E \mid y(t) > 0, \ t \in [\mu_1, \mu_2] \}, \quad \partial M = \{ y \in M \mid y(t_0) = 0, \ \text{for some} \ t_0 \in [\mu_1, \mu_2] \},$$

$$Q^\circ = \{ y \in E \mid y(t) > 0, \ t \in [\tau, 1] \}, \quad \partial Q = \{ y \in Q \mid y(t_0) = 0, \ \text{for some} \ t_0 \in [\tau, 1] \}.$$
Let the functionals $\alpha$ and $\beta$ be defined by

$$\beta(y) = y(\mu_2) \quad \text{and} \quad \alpha(y) = y(\tau).$$

Note that $\alpha$ and $\beta$ are both linear (hence both $\alpha$ and $\beta$ are concave and convex functionals), continuous, nonnegative functionals.

Claim 1: $T$ is LW-inward with respect to $P_Q(B, \alpha, y_B, a)$.

Subclaim 1.1: $\{y \in P : a < \alpha(y) \quad \text{and} \quad B(y) \ll_Q y_B\} \neq \emptyset$.

Define $y_0 \in P$ by

$$y_0 = \begin{cases} \frac{a + y_B(\tau)}{y(\tau)} & \text{for } t \in [0, \tau], \\ \frac{a + y_B(1)}{y(1)} & \text{for } t \in [\tau, 1]. \end{cases}$$

Note that $y_B(\tau) \geq \tau y_B(1)$, $a < y_B(1)\tau$,

$$\alpha(y_0) = y_0(\tau) = \frac{a + y_B(\tau)}{2} \geq \frac{a + y_B(1)\tau}{2} > a,$$

and

$$(B y_0)(t) = y_0(t) = y_0(\tau) = \frac{a + y_B(\tau)}{2} \leq \frac{y_B(1)\tau + y_B(\tau)}{2} \leq y_B(\tau) \leq y_B(t), \quad t \in [\tau, 1];$$

that is, $a < \alpha(y_0)$ and $B(y_0) \ll_Q y_B$. Hence, $y_0 \in \{y \in P : a < \alpha(y) \quad \text{and} \quad B(y) \ll_Q y_B\}$.

Subclaim 1.2: If $y \in \partial P_Q(B, y_B)$ and $a \leq \alpha(y)$, then $B(Ty) \ll_Q y_B$.

Let $y \in \partial P_Q(B, y_B)$ and $a \leq \alpha(y)$. Then $a \leq y(\tau)$, $y(t) \leq y_B(t)$ for $t \in [\tau, 1]$, and there exists $t_0 \in [\tau, 1]$ such that $y(t_0) = y_B(t_0)$. Hence, for $t \in [0, \tau]$, by the concavity (3.3) of $y$

$$a \frac{t}{\tau} \leq y(t) \leq y_B(\tau),$$

and by using the monotonicity of $y$ with $a \leq y(\tau)$ and $y(t) \leq y_B(t)$ for $t \in [\tau, 1]$ we have

$$a \leq y(\tau) \leq y(t) \leq y_B(t) \quad \text{for } t \in [\tau, 1].$$

If $f$ satisfies (i), then for all $t \in [\tau, 1]$,

$$(BT y)(t) = (Ty)(t) = \int_0^1 G(t, s) f(y(s)) ds$$

$$= \int_0^\tau sf(y(s)) ds + \int_\tau^t sf(y(s)) ds + \int_0^1 tf(y(s)) ds$$

$$\leq \int_0^\tau sf(y(s)) ds + \int_\tau^t tf(y(s)) ds$$

$$< y_B(t) - t \int_\tau^1 f(y_B(s)) ds$$

$$= y_B(t).$$
If $f$ satisfies (i'), then for all $t \in [\tau, 1]$,

\[
(BTy)(t) \leq \int_0^\tau s f(y(s))ds + \int_\tau^1 t f(y(s))ds
\]

\[
< \int_0^\tau s \left( \frac{a}{\tau} \right) ds + \frac{y_B(t) - \int_0^\tau sf(\frac{a}{\tau} s)ds}{t(1-\tau)} \int_\tau^1 ds = y_B(t).
\]

Therefore, $B(Ty) \ll y_B$.

Subclaim 1.3: If $y \in \partial P_Q(B, y_B)$ and $\alpha(Ty) < a$, then $B(Ty) \ll y_B$.

Let $y \in \partial P_Q(B, y_B)$ and $\alpha(Ty) < a$. Then,

\[
\alpha(Ty) = (Ty)(\tau) < a < y_B(1) \tau,
\]

and for $t \in [\tau, 1]$,

\[
(BTy)(t) = (Ty)(t) = \int_0^1 G(t, s)f(y(s))ds
\]

\[
< \frac{t}{\tau} \int_0^1 G(\tau, s)f(y(s))ds = \frac{t}{\tau}(Ty)(\tau)
\]

\[
< ty_B(1) = y_B(t).
\]

Therefore, $B(Ty) \ll y_B$.

Subclaim 1.4: $P_Q(B, y_B)$ is bounded.

Let $y \in P_Q(B, y_B)$. Then, $(By)(t) = y(t) \leq y_B(t)$ for $t \in [\tau, 1]$. Hence, $y(t) \leq y_B(1)$ for $t \in [0, 1]$; that is, $P_Q(B, y_B)$ is bounded.

Claim 2: $T$ is LW-outward with respect to $P_M(\beta, A, b, y_A)$.

Subclaim 2.1: $\{y \in P : y_A \ll_M A(y), \beta(y) < b\} \neq \emptyset$.

Let $y_0(t) = \frac{1}{2} \left( \frac{y_A(\mu_1)}{\mu_1} + \frac{b}{\mu_2} \right) t$. Then $y_0 \in P$ and since $\frac{y_A(\mu_1)}{\mu_1} \mu_2 < b$, we have

\[
\beta(y_0) = y_0(\mu_2) = \frac{1}{2} \left( \frac{y_A(\mu_1)}{\mu_1} \mu_2 + b \right) < b,
\]

and

\[
(Ay_0)(t) = y_0(t) > \frac{y_A(\mu_1)}{\mu_1} t \geq y_A(t), \text{ for all } t \in [\mu_1, \mu_2].
\]

Hence, $y_0 \in \{y \in P : y_A \ll_M A(y), \beta(y) < b\}$.

Subclaim 2.2: If $y \in \partial P_M(A, y_A)$ and $\beta(y) \leq b$, then $y_A \ll_M A(Ty)$.

Let $y \in \partial P_M(A, y_A)$ with $\beta(y) \leq b$. Then, $y(t) \leq y_A(t)$ for $t \in [\mu_1, \mu_2]$, $y(t_0) = y_A(t_0)$ for some $t_0 \in [\mu_1, \mu_2]$, and $y(\mu_2) \leq b$. Hence, for $t \in [0, \mu_2]$ by the monotonocity and concavity
of $y$ and $y_A$ we have

$$\frac{y_A(\mu_1)}{\mu_2} t \leq \frac{y_A(t_0)}{\mu_2} t = \frac{y(t_0)}{\mu_2} t \leq \frac{y(\mu_2)}{\mu_2} t \leq y(t) \leq y_A(\mu_2),$$

and similarly for $t \in [\mu_2, 1]$

$$y_A(\mu_1) \leq y_A(t_0) = y(t_0) \leq y(t) \leq \frac{y(\mu_2)}{\mu_2} t \leq \frac{y_A(\mu_2)}{\mu_2} t.$$

If $f$ satisfies (ii), then for $t \in [\mu_1, \mu_2]$,

$$(ATy)(t) = (T(y)(t) = \int_0^1 G(t, s)f(y(s))ds$$

$$\geq \int_{\mu_2}^t tf(y(s))ds > \frac{y_A(t)}{t(1 - \mu_2)} \int_{\mu_2}^t tds = y_A(t).$$

On the other hand, if $f$ satisfies (ii') and if we note that \(t^2 + \frac{3}{2}t(\mu_2^2 - t^2)\) is increasing over \([\mu_1, \mu_2]\), then for $t \in [\mu_1, \mu_2]$,

$$(ATy)(t) = (Ty)(t) = \int_0^1 G(t, s)f(y(s))ds$$

$$\geq \int_0^t sf(y(s))ds + \int_{\mu_2}^{\mu_2} tf(y(s))ds$$

$$\geq \frac{6\mu_2}{(3\mu_2^2 - \mu_1^2)\mu_1} \left( \int_0^t sy(s)ds + \int_1^{\mu_2} ty(s)ds \right)$$

$$\geq \frac{6\mu_2}{(3\mu_2^2 - \mu_1^2)\mu_1} \frac{y_A(\mu_1)}{\mu_2} \left( \int_0^t s^2ds + \int_1^{\mu_2} t^2ds \right)$$

$$\geq \frac{6\mu_2}{(3\mu_2^2 - \mu_1^2)\mu_1} \frac{\mu_1 y_A(\mu_2)}{\mu_2} \left( \frac{t^3}{3} + \frac{1}{2}t(\mu_2^2 - t^2) \right)$$

$$\geq \frac{6\mu_2}{(3\mu_2^2 - \mu_1^2)\mu_1} \frac{\mu_1 y_A(\mu_2)}{\mu_2} \frac{(3\mu_2^2 - \mu_1^2)\mu_1}{6} = y_A(\mu_2)$$

$$\geq y_A(t).$$

Then, $y_A \leq M A(Ty)$.

Subclaim 2.3: If $y \in \partial P_M(A, y_A)$ and $b < \beta(Ty)$, then $y_A \leq_M A(Ty)$.

Let $y \in \partial P_M(A, y_A)$ and $b < \beta(Ty)$. Then, $(Ty)(\mu_2) > b$, and so for $t \in [\mu_1, \mu_2]$,

$$(ATy)(t) = (Ty)(t) = \int_0^1 G(t, s)f(y(s))ds$$

$$\geq \frac{t}{\mu_2} \int_0^1 G(\mu_2, s)f(y(s))ds$$

$$\geq \frac{t}{\mu_2} b > \frac{t}{\mu_2} \cdot \frac{y_A(\mu_1)}{\mu_1} \cdot \mu_2 \geq \frac{y_A(\mu_1)}{\mu_1} t \geq y_A(t),$$
that is, $y_A \preceq_M A(Ty)$.

Subclaim 2.4: $P_M(A, y_A)$ is bounded.

Let $y \in P_M(A, y_A)$. Then $y(t) \leq y_A(t)$ for $t \in [\mu_1, \mu_2]$.

For $t \in [\mu_2, 1]$, $y(t) \leq \frac{1}{\mu_2} y(\mu_2) \leq \frac{y_A(\mu_2)}{\mu_2}$. Hence, $y$ is bounded by $\frac{y_A(\mu_2)}{\mu_2}$.

Claim 3: If $y_A(\mu_1) < y_B(1) \min \{\tau, \mu_1\}$, then $P_M(A, y_A) \subseteq P_Q(B, y_B)$; if $y_B(1) < y_A(\mu_1)$, then $P_Q(B, y_B) \subseteq P_M(A, y_A)$.

Suppose $y_A(\mu_1) < y_B(1) \min \{\tau, \mu_1\}$, then for $y \in P_M(A, y_A)$ and $t \in [\tau, 1]$,

$$y(t) \leq \frac{t}{\tau} y(\tau) \leq \frac{t}{\tau} \max \left\{ \frac{y(\mu_1)}{\mu_1}, \frac{\tau y(\mu_1)}{\mu_1} \right\} \leq \frac{t}{\tau} \max \left\{ y_A(\mu_1), \frac{\tau y_A(\mu_1)}{\mu_1} \right\}$$

$$< \frac{t}{\tau} \max \left\{ y_B(1) \min \{\tau, \mu_1\}, \frac{\mu_1}{\tau} y_B(1) \min \{\tau, \mu_1\} \right\}$$

$$\leq \frac{t y_B(1)}{\tau} \max \{\min \{\tau, \mu_1\}, \tau\} = t y_B(1) \leq y_B(t);$$

that is, $P_M(A, y_A) \subseteq P_Q(B, y_B)$.

Define $y_0(t) = \frac{1}{2}(y_A(t) + y_B(t))$. Since $y_A \in P_M(A, y_A)$, $y_A \in P_Q(B, y_B)$. Therefore, $y_0 \in P_Q(B, y_B)$ since $P_Q(B, y_B)$ is convex. Notice

$$y_0(\mu_1) = \frac{1}{2}(y_A(\mu_1) + y_B(\mu_1)) \geq \frac{1}{2}(y_A(\mu_1) + \mu_1 y_B(1))$$

$$> \frac{1}{2}(y_A(\mu_1) + y_A(\mu_1)) = y_A(\mu_1),$$

and so, $y_0 \notin P_M(A, y_A)$; that is, $P_M(A, y_A) \not\subseteq P_Q(B, y_B)$.

Suppose $y_B(1) < y_A(\mu_1)$. Let $y \in P_Q(B, y_B)$. Then $y(t) \leq y_B(t)$ for $t \in [\tau, 1]$. Since $y_B(1) < y_A(\mu_1)$, so $y(t) \leq y_B(1) < y_A(\mu_1) \leq y_A(t)$ for $t \in [\mu_1, \mu_2]$. Hence, $y \in P_M(A, y_A)$. Define $y_0(t) = \frac{1}{2}(y_A(t) + y_B(t))$. Then $y_0 \in P$ since $y_A, y_B \in P$ and $P$ is convex. Notice for $t \in [\mu_1, \mu_2]$, $y_0(t) < \frac{1}{2}(y_A(t) + y_A(\mu_1)) \leq \frac{1}{2}(y_A(t) + y_A(t)) = y_A(t)$ and $y_0(1) = \frac{1}{2}(y_A(1) + y_B(1)) > \frac{1}{2}(y_A(\mu_1) + y_B(1)) > y_B(1)$, and so $y_0 \in P_M(A, y_A) - P_Q(B, y_B)$, i.e., $P_Q(B, y_B) \subseteq P_M(A, y_A)$.

Therefore, if $y_A(\mu_1) < y_B(1) \min \{\tau, \mu_1\}$, then by (H1) of Theorem 2.8, $T$ has a fixed point $y^* \in P_B(A, y_A, y_B, Q, M)$; that is, (3.1). (3.2) has a solution $y^* \in P$ such that $y^*(t) < y_B(t)$ for $t \in [\tau, 1]$, and $y^* \notin y_A(t)$ for $t \in [\mu_1, \mu_2]$.

Moreover, if $y_B(1) < y_A(\mu_1)$, then by (H2) of Theorem 2.8, $T$ has a fixed point $y^{**} \in P(A, B, y_A, y_B, M, Q)$. So, (3.1), (3.2) has a solution $y^{**} \in P$ such that $y^{**}(t) < y_A(t)$ for $t \in [\mu_1, \mu_2]$, and $y^{**}(t) \notin y_B(t)$ for $t \in [\tau, 1]$.

\[\square\]

Remark. The functions $y_A$ and $y_B$ in Theorem 3.1 are possible concave nondecreasing nonnegative functions that might be used as functional boundaries for positive solutions of (3.1), (3.2).
Example. Consider the right focal boundary value problem,
\[ x''(t) + 1 - \sin x = 0, \quad t \in (0, 1), \]
\[ x(0) = x'(1) = 0. \]

We would like to choose some quadratic functionals as the boundaries for our positive solution. In that direction, let \( \mu_1 = \frac{3}{4}, \mu_2 = 1, \tau = \frac{1}{2}, y_A(t) = \frac{\pi}{20} t (2 - t), y_B(t) = \frac{\pi}{5} t (2 - t), \]
\[ a = \frac{\pi}{8}, b = \frac{\pi}{15}. \]

Then \((i'), (ii'), \) and \( y_A(\mu_1) < y_B(1) \min \{ \tau, \mu_1 \} \) are satisfied in Theorem 3.1 for the above problem. Therefore the above boundary value problem has at least one positive solution \( y^* \) with \( y^*(t) < \frac{\pi}{5} t (2 - t) \) for \( t \in \left[ \frac{1}{2}, 1 \right], \) and \( y^*(t) \not\in \frac{\pi}{20} t (2 - t) \) for \( t \in \left[ \frac{3}{4}, 1 \right]. \)

In comparison, consider the following representative example of an application of a functional fixed point theorem.

Example: The boundary value problem
\[ x''(t) + \frac{1}{\sqrt{x}} + e^{x-2} = 0, \]
with right-focal boundary conditions
\[ x(0) = 0 = x'(1), \]
has at least one positive solution \( x^* \) which was verified by the Functional Expansion - Compression Fixed Point Theorem of Leggett-Williams Type in [8] by Anderson, Avery and Henderson (let \( b = 1, \ c = 5, \) and \( \tau = \frac{1}{2} \)). The positive solution \( x^* \) has the properties
\[ 1 \leq x^*(1) \quad \text{and} \quad x^*(\tau) \leq 5. \]

Remark. Note the difference between the global knowledge and the local knowledge of the solutions obtained in the previous two examples by applying the operator type fixed point theorem instead of the functional type fixed point theorem. Existence of solutions arguments involving operators provide global knowledge, thus moving existence of solutions arguments in the direction of numerical arguments which adds value to existence of solutions arguments.

References


