CHAPTER 4

Mathematics and the Divine in Plato

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MATHEMATICS AND THE DIVINE: A HISTORICAL STUDY
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"*God is always doing geometry*"

1. Preliminary remarks

Plato (ca. 429–347 BCE) of Athens is known primarily as a philosopher.[1] It is clear that he had an intense admiration for mathematics and that his notion of philosophical method was based on reflection on mathematical reasoning.[2] His frequent references to and discussions of mathematics in his dialogues played a major role in establishing the idea that mathematics is of central human importance, and his making five branches of study (*mathêmata*), arithmetic, plane and solid geometry, astronomy, and harmonics, central to his notion of higher education in Book VII of the *Republic* is the source for the *quadrivium*, a fundamental component of later Western education. It is also very likely that Plato played a significant role in the rapid development of Greek mathematics in the fourth century BCE.[3] On the other hand, it is very unlikely that Plato made substantive contributions to mathematics;[4] indeed, many of the more specifically mathematical passages in his works have no clear and correct interpretation, and many of them can be read as the half-understandings of an enthusiastic spectator.

In the twentieth century much work has been done in trying to determine the chronological order in which Plato wrote his dialogues.[5] In this essay I am going to concentrate on two dialogues, which are a cornerstone of a traditional interpretation of Plato's philosophy and in which he gives his most influential treatments and uses of mathematics, namely the *Republic*, usually classed as a mature dialogue, and the *Timaeus*, usually classed as a late one.[6] We will see that there are some, at least apparent, tensions in the use of mathematics in the two dialogues. However, there is a much greater tension between the ways in which the two dialogues represent the divine,[7] even though they agree in their disregard for what might be called traditional Greek religion.

Explaining this tension requires me to bring in another aspect of Plato's philosophical outlook common to the *Timaeus* and the *Republic* and thought to be a linchpin of Plato's mature philosophy: the so-called theory of forms. The scholarly literature on this topic is both enormous and contentious, and I have no intention of pursuing interpretive issues here.[8] For purposes of this paper it suffices to think of forms as eternal, intelligible objects (like real numbers or functions as opposed to perceptible objects like tables or horses) corresponding to general terms such as 'just', 'large', 'triangle', 'fire', 'horse'. We might now refer to these objects as concepts, but it is important to realize that for Plato forms are real objects, indeed more real than things like tables or horses. For Plato the forms constitute

[1] On Plato in general I mention two collections of essays, [2] and [16].

[2] See [18].

[3] Cp. [17].

[4] On this topic see [7].

[5] See, for example, [4].

[6] There was much discussion of the place of *Timaeus* after the attempt by Owen [21] to group the *Timaeus* with the *Republic* as a mature work. But it seems that scholarly consensus now backs the response to Owen by Cherniss [8].

[7] On the concept of divinity in Plato see [25] and [11].

[8] Many of the articles in [2] discuss the forms.

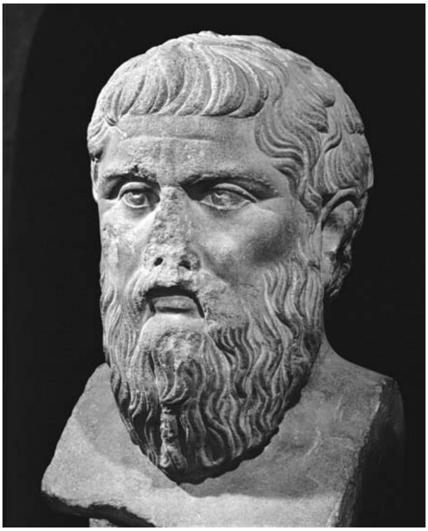

Fig. 1. Plato. Marble bust, Roman copy of an original dated 4th BCE. Height 43 cm. Louvre, Paris. Photo: Erich Lessing.

some kind of system, of which it is the task of the philosopher to gain an understanding. On the whole the knowledge of forms is conceptual knowledge. Philosophers who know the form corresponding to the term 'just' will have the answer to the question 'What is justice' and will be able to assess the justice of particular actions, programs, etc., things which are said to be just by participating or sharing in the form of justice.

In the *Timaeus* Plato describes the creation of our world or cosmos by a benevolent creator god (standardly referred to as a craftsman (*dêmiourgos*, frequently rendered as demiurge) or maker, but also called mind or reason (*nous*)), who tries to make the world as good as possible by making it resemble the forms as much as possible. This conception of a god as an intelligence making the world is found in other dialogues considered to be late, explicitly in the *Philebus* and with some complications in the *Laws*.[9] But it is not found in any explicit way[10] in the *Republic* or in other earlier dialogues. In the *Republic* Plato makes the distinction between forms and perceptible things the basis of a distinction between knowledge and mere belief or opinion. He makes knowledge of the forms the basis of the philosopher's superiority to other human beings, and he makes the form of the good, which he treats as the unhypothetical first principle of all things, the pinnacle of the system of the forms. The forms themselves are treated as objects of rational desire, understanding of which provides the fullest satisfaction. Plato is willing to speak of people with knowledge of the forms as divine, and it is quite clear that he thinks of the forms as divine as well. In this sense the "gods" of the *Republic* are the forms and the highest "god" is the form of the good. In the *Timaeus* mathematics is related to the divine because the

[9] See [15] or [11, pp. 186–191].

[10] At *Republic* 530a there is a reference to the demiurge of heaven, but that it presumably because a point is being made about heaven by reference to the products of craftsmen.

demiurge uses mathematics in fashioning the world; in the *Republic* mathematics is related to the divine because knowledge of it is an important step on the pathway to knowing forms.

Many attempts have been made to reconcile these apparently discrepant conceptions of divinity in the *Timaeus* and the *Republic*. The developmentalist alternative of saying that Plato came to believe in a creating god or mind in his later years is always available. The other most influential position, which has an ancient pedigree, involves the claim that the *Timaeus* account of creation is a narrative representation of eternal truths about an eternal world which reveals its composition, so that the demiurge can be treated as a fictional device. I confess that I am unhappy with either interpretation but unable to provide a satisfactory alternative. This unfortunate situation is, I think, ameliorated by the fact that the representations of the relationship of mathematics to the divine are quite parallel in the two dialogues. I shall then focus on that relationship and say little more about the intrinsic character of the divine in the two dialogues.

Before turning to the body of my paper I want to mention one set of texts other than Plato's dialogues which are sometimes brought to bear in discussions of the role of mathematics in his philosophy. Aristotle has many remarks, almost all critical, of positions taken by Plato and his associates. In a disconcerting number of cases the positions have no clear relation to anything we find in Plato's dialogues. Other later writers also mention these positions and discuss them in more detail than Aristotle does, although we usually do not know what their sources of information are. Many scholars react to this situation by referring to Plato's unwritten doctrines, and many of them think of the doctrines as ideas developed, probably very schematically, by Plato late in his life. There is, however, a group of philosophers who think of the unwritten doctrines or, perhaps now better, doctrine as a relatively worked out theory which in some sense is the core of Platonic philosophy and underlies in one way or another what is said in the dialogues. Since mathematical ideas play an important role in this interpretation, I shall make occasional references to the notion of unwritten doctrines, but I shall not go into this topic in any detail.[11]

2. Introduction

In one of the dialogues included in Plutarch's "Table-Talk"[12] the topic of conversation is what Plato meant when he made the statement I have used as an epigraph. Plutarch admits that there is no clear evidence that Plato ever did say this, but says that it is believable that he made the statement, which is in conformity with Plato's nature. The speakers offer four accounts of Plato's meaning which are more like accounts of the importance Plato attaches to mathematics. The first clearly reflects the role assigned to mathematics in the *Republic*:

1. Geometry turns us away from perception and towards the intelligible, eternal nature, contemplation of which is the goal of philosophy.

[11] On the unwritten doctrines see [12] and [13], and the collection of essays in the Argentinian journal *Methexis* 6 (1993). Ref. [22] gives a useful presentation of Aristotle's problematic descriptions of Plato's views.

[12] VIII.2 *Pôs Platôn elege ton theon aei geômetrein?*

The second, which is of less importance, has a relation to the *Republic*, but it picks up specifically on things said by Plato in the *Laws* and the *Gorgias*:

2. It is better to distribute goods on the basis of geometric than arithmetic proportion, since with the latter goods are distributed equally, whereas with the former they are distributed according to merit;[13]

The third and fourth relate to Plato's *Timaeus*, the dialogue I shall discuss first:

3. In creating the world god imposed limit on an unlimited matter, using geometrical shapes;
4. In the *Timaeus* Plato distinguished three principles which we call god, matter, and form; in making and maintaining the cosmos, god imposes form on the entire quantity of matter, and this is like solving that central geometric problem (Euclid, *Elements* VI, 25) of constructing a geometric figure equal to a given one and similar to another.

3. The *Timaeus*[14]

I have already mentioned that the *Timaeus*, which is named after the character who gives the discourse constituting most of the dialogue, is a description of the creation of the world by a benevolent god who strives to make the world as like the forms as possible. The most important form for god's creative activity is the form of living thing (*zôion*), a form including the forms of all living things and in imitation of which god creates the cosmos, "a single, visible living thing, containing within itself all living things ..." (30d–31a). The conception of the cosmos as in some way alive dominates ancient thinking about our world and separates it from modern physics, which purports to give a universal description of Plato's sensible world without making any specific reference to the fact of life, the domain of biology, or to intelligence, the domain of psychology. For Plato the cosmos is alive, and (by definition) that means it has a soul; for Plato, it also has a mind and intelligence (*nous*). It is difficult to spell out with any precision the exact consequences of this difference between the Platonic conception of the world as alive and the tendency of modern physics to abstract from the fact of life, but I think the most important related fact, if not consequence, is that Plato does not use mathematics for the formulation of laws on the basis of which experiments can be performed, measurements made, and outcomes predicted. For Plato the mathematical character of the world is most importantly a sign of its intelligent organization (and perhaps of the intelligence of its organizer) and—more or less indistinguishably—of its goodness and beauty.

[13] The conception involved here is difficult to formulate rigorously. But the idea is that in a geometric proportion v_1 is to m_1 as v_2 is to m_2, so that if v_1 and v_2 represent the values of goods g_1 and g_2 and m_1 and m_2 the merits of two people p_1 and p_2, then it will be just to assign g_1 to p_1 and g_2 to p_2; on the other hand if, g_1, g, and g_2 are in arithmetic proportion, then g is simply half of g_1 and g_2, and assigning goods on the basis of arithmetic proportion to p_1 and p_2 will allegedly be unjust because the two unequal people are given equal amounts.

[14] The standard translation and commentary for the *Timaeus* is [10]. I have used this translation with some revisions. Another very useful translation and commentary is [5].

3.1. *Why there are four "elements", earth, water, air, and fire*[15] *(Timaeus 31b–32b)*

Timaeus' first specific invocation of mathematics comes in his first description of the generation of the body of the world. I shall go into some detail on this passage because it is illustrative of significant difficulties in interpreting the way Plato uses mathematics. In outline Timaeus proceeds as follows:

(a) What has a body must be visible and tangible.
(b) Nothing can be visible without fire.
(c) Nothing can be tangible without something solid (*stereos*).
(d) Nothing is solid without earth.
(e_1) "But two things cannot be well united without a third; for there must be some bond between them drawing them together.
(e_2) And of all bonds the best is that which makes itself and the terms it connects one in the fullest sense; and it is of the nature of a proportion to effect this most perfectly.
(e_3) For whenever of three numbers $\langle a, b, c\rangle$ the middle one between any two that are either bulks (*ongkoi*) or powers (*dunameis*)[16] is such that "a is to b as b is to c and c is to b as b is to a, "then, since the middle becomes first and last and again the last and first become middle $\langle$i.e., b is to a as c is to $b\rangle$, in that way all will necessarily turn out to be the same thing and therefore they will all be one."
(f) If the body of the universe were a plane without depth, one mean would be sufficient, but it is a solid and solids are fit together by two, not one, mean;
(g) Therefore god made water and air means between fire and earth, and contrived it so that fire is to air as air is to water as water is to earth.

It is clear enough that some of the premisses, e.g., (b), (d), (e_1), are questionable and apparently *ad hoc*. But even if we grant all of the premisses which we understand, there remain premisses and a conclusion the meaning of which is not transparent. In what sense are 4, 6, and 9 one because 4 is to 6 as 6 is to 9? Timaeus' invocation of the possible positions of terms in a three-term proportion hardly seems an adequate answer, but the alleged unifying nature of proportion is clearly an important part of Plato's mathematical vision of the cosmos. And the way he invokes proportion suggests that he does not understand proportionality as mere proportionality in the way in which we might read an algebraic equation as merely an equation. Rather he reads a proportionality among entities as a kind of "force" for bringing things together, not simply as a way of describing relations among things.

A perhaps more important point concerns the whole structure of the reasoning. From our perspective its subject is physical reality, earth, water, air, and fire, out of which the universe is made. These are to be blended into a unified whole, but why should we think the simple proportionality of (g) will produce that unity? Moreover, what is the proportionality of (g) a proportionality of? The volumes of the "elements"? Their weights? Again Timaeus does not say, and it seems unlikely that Plato had specific quantities in mind. For him the picture

[15] I retain the standard appellation 'elements' for earth, water, air, and fire, but I put 'elements' in quotation marks because, as we shall see at the beginning of Section 3.3, Timaeus specifically denies that these 'elements' really are elements (48b–c).

[16] These words are not standard arithmetical or, more generally, mathematical terms, and their purpose is not understood. Dropping them from the assertion would not affect the truth—or falsity—of the claim that proportionality produces unity.

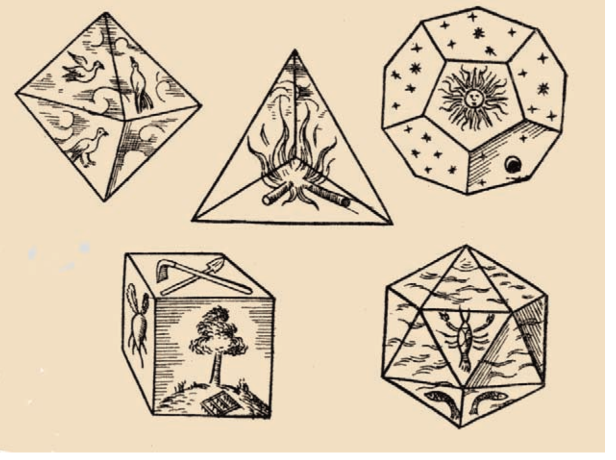

Fig. 2. The five (regular) Platonic solids as depicted in Johannes Kepler's *Harmonices Mundi*, 1619.

of there being something mathematical and relatively simple is an adequate support for the idea that the god's cosmos is good.

Finally there is the question of the apparently mathematical assertion (f). It seems unlikely that Plato was here thinking of some geometric truth.[17] He may have had in mind some version of what we know as propositions 18 to 21 of book VIII of Euclid's *Elements*:

- m and n are similar plane numbers if and only if there is one mean proportional number between m and n;
- m and n are similar solid numbers if and only if there are two mean proportional numbers between m and n.[18]

There is obviously a parallelism between these arithmetic truths and Timaeus' claims about the four "elements", but it is difficult to see any more specific relation between them. Plato moves from an abstract mathematical proposition about the existence of means to a claim about the make-up of the physical world without explaining how he passes from one to the other. He does not purport to look at the physical world and discover a mathematical

[17] If, for example, (f) presupposes that if there are two solids of volumes v_1 and v_3 there is no solid of volume v_2 where v_1 is to v_2 as v_2 is to v_3, then (f) presupposes something which is false from our perspective. We can refine the issue by introducing some notion of constructing a solid, and we can restrict the possible solids to convex ones contained by rectilinear planes, but it is not true either that a solid which is a mean between two such solids cannot sometimes be constructed or that between two such solids two mean solids always can be constructed. Of course, there are many other ways of trying to extract some geometric truth from (f), but I do not know of any satisfactory one.

[18] m and n are similar plane numbers if and only if, for some m_1, m_2, n_1, n_2, $m = m_1 \cdot m_2$ and $n = n_1 \cdot n_2$, and m_1 is to m_2 as n_1 is to n_2 (or equivalently, m_1 is to n_1 as m_2 is to n_2). Consequently, if m and n are similar plane numbers, $m_1 \cdot m_2$ is to $n_1 \cdot m_2$ as $n_1 \cdot m_2$ is to $n_1 \cdot n_2$, and $n_1 \cdot m_2$ is a mean proportional between m and n. On the other hand, suppose m is to l as l is to n; let m' and l' be the least numbers such that m' is to l' as m is to l. Then, for some j and j', $m = j \cdot m'$, $l = j \cdot l' = j' \cdot m'$, and $n = j' \cdot l'$, and j is to j' as $j \cdot m'$ is to $j' \cdot m'$ as $j' \cdot m'$ is to $j' \cdot l'$ as m' is to l', and m and n are similar plane numbers.
The treatment of similar solid numbers is analogous. However, it should be noted that the same numbers, e.g., 2^6 and 3^6, can be similar plane and similar solid numbers.

relationship among its elements. Rather he takes an alleged truth about proportionality and the fact that he is dealing with three-dimensional visible entities and infers (or at least makes a claim about) what the relationship of the "elements" of the cosmos is. Of course, I am not claiming that Plato actually came to the conclusion that earth, water, air, and fire are basic components of the world as a result of this kind of thinking. He undoubtedly took the idea over from other physicists, notably Empedocles. But he chooses to present the conclusion as a development of a mathematical proposition, and that kind of presentation proved to be extremely influential in later philosophy.

3.2. *The construction of the soul of the world (Timaeus 35a–36d)*

Plato's presentation of the construction of the soul of the cosmos has the same *a priori* character. Timaeus imagines the demiurge producing a blend[19] of certain forms and then a division of this into portions. The ratios of the proportions, in fact, correspond to the standard diatonic scale of Greek mathematical harmonics, but Timaeus never mentions this fact; he simply gives an opaque series of arithmetic relations in terms of which the demiurge makes the division.[20] One might think that Plato was playing with mathematics and with the reader.

Once the division has been made, it drops from sight and the soul stuff is treated as a unified whole. This is divided into two equal strips which, in turn, are made into concentric circles set at an angle to one another and rotating in opposite directions, the "inner" circle itself being divided into seven unequal circles. Timaeus gives enough of a description of the motions of these circles to make clear that what underlies his account is a rough, geocentric, astronomical model in which the "outer" circle represents the sphere of the fixed stars, the inner circles sun, moon, and the five planets known at that time. But that this is so is only made clear some two pages later when Timaeus' mentions some of the heavenly bodies explicitly.[21] So again mathematics comes into the *Timaeus* in the service of a kind of mathematical physics, in this case astronomy. But the mathematics is not presented as rising out of the physics, but is rather introduced in an apparently abstract way, the motivation of which is completely obscured.

3.3. *The geometry of the "elements" (Timaeus 48b–57c)*

Some ten pages after the discussion of the construction of the world soul Timaeus announces that, whereas he has been mainly discussing the creative activity of mind, he now has to bring in what he calls necessity and make in effect a new beginning:

[19] On a literal reading the whole description treats the soul as if it were a compound of extended magnitudes, a fact which Aristotle criticized as mistaken in the case of soul (*On the Soul*, I.3.406b26–407b11) because it is not an extended magnitude. Later Platonists insisted that Aristotle was reading a "mythological" passage too literally.

[20] For a more detailed discussion of this passage see the appendix.

[21] When the astronomical significance of the final arrangement of the soul stuff is put together with the musical ideas underlying its first division into portions, it is difficult to avoid the conclusion that some doctrine of heavenly music (the "harmony of the spheres") lies behind what Timaeus is made to say. But it is not to be found explicitly in what Timaeus does say. (On the harmony of the spheres see [6, pp. 350–357].)

> We must, in fact, consider in itself the nature and properties of fire, water, air, and earth before heaven came to be. For to this day no one has explained their genesis, but we speak as if men knew what fire and each of the others is, positing them as elements or letters[22] of the universe. But one who has ever so little intelligence should not rank them in this analogy even so low as syllables. On this occasion, however, our contribution is to be limited as follows. We are not to speak of the first principle or principles—or whatever name men choose to employ—of all things, if only on account of the difficulty of explaining what we think by our present method of exposition. (48b–c)

After a few more words Timaeus launches into his new beginning and brings us into the third interpretation of the epigraph to this paper.

Timaeus begins by adding to the distinction between the eternal world of forms and its perceptible copy a third item which he calls (among other things) the receptacle and which later Platonists usually called matter. He compares the receptacle to a mass of soft gold which can be molded into a variety of shapes, but in itself has no particular shape; so too the receptacle receives the likeness of the eternal forms although in itself it has no particular character. "It is invisible and without form; it receives everything; and it participates in what is intelligible in a way which is puzzling and difficult to grasp" (51a–b). Timaeus goes on to describe a pre-cosmic situation in which the receptacle shakes and is shaken by rudimentary versions of the four "elements" in a wild way.

> Fire, water, earth, and air possessed indeed some vestiges of their own nature, but were altogether in such a condition as we should expect for anything when deity is absent from it. Such being their nature at the time when the ordering of the universe was taken in hand, the god began by shaping them by means of ⟨geometric⟩ forms[23] and numbers. That the god shaped them with the greatest possible perfection, which they had not before, must be taken, above all, as a principle we constantly assert; what I must now attempt to explain to you is the arrangement and genesis of each of them. The account will be unfamiliar, but you are schooled in those branches of learning ⟨i.e., mathematics⟩ which my explanations require, and so will follow me. (53b–c)

Timaeus now makes a quick series of mathematical moves when he asserts:

(a) The four "elements" are bodies, and bodies are three-dimensional.
(b) What is three-dimensional is contained by planes.[24]
(c) Rectilinear plane surfaces are divisible into triangles.
(d) All triangles are divisible into right triangles, whether isosceles or scalene.

Timaeus says he will hypothesize the right-angled triangles as the *archê* of the simple bodies, but, as he had done before at 48b–c, he announces that he is not going to talk about ultimate principles, ones known to god and anyone who might qualify as god's friend. Shortly after this Timaeus points out that, whereas all isosceles right triangles have the same "nature", presumably because they are all similar, there are infinitely many natures for the scalene right triangles of which he proposes to choose the most beautiful (*kallistos*), the half-equilateral. After saying that it would take too long to explain this choice, he opts for the triangle with sides in the ratio of 1, 2, and $\sqrt{3}$. I shall call these two kinds of triangles the rudimentary triangles.

[22] The Greek word for a letter of the alphabet is *stoikheion*, a word which came to be applied to elements or fundamental building blocks of the world.

[23] Timaeus speaks only of forms (*eidê*), which could mean Platonic forms, but the likelihood is that here he means geometric forms or shapes. In any case, so I shall assume. Cp. [5, p. 252, n. 386].

[24] This is obviously not true of all bodies, but only of those in which Timaeus is interested.

Just before selecting these two triangles as principles Timaeus says that his task is to find the four most perfect bodies "such that some can come to be from one another by dissolution" (53e2), a demand whose meaning becomes clear only when the task has been done, and the four simple bodies have been assigned to four of the five regular solids which Euclid constructs in the last book of the *Elements*:[25]

- the cube contained by 6 squares, assigned to earth;
- the triangular pyramid contained by 4 equilateral triangles, assigned to fire;
- the octahedron contained by 8 equilateral triangles, assigned to air;
- the icosahedron contained by 20 equilateral triangles, assigned to water;

Timaeus' job is to somehow reduce these four solids to his principles, the two rudimentary triangles. He does this by dividing the faces of the regular solids into rudimentary triangles and then indicating how the faces can be put together to form the solids.[26] The construction of the solids out of their faces is the basis of Timaeus' account of the way in which "elements" change into one another, which is a matter of these invisibly small solids breaking down into their faces and these faces recombining into different solids. This means that although water, air, and fire can be transformed into one another, earth is never involved in elemental change. When it is broken into its faces the faces can only reassemble into another cube (56d).[27]

There are a number of things to be said about this whole remarkable construction. Let us begin with an issue which Timaeus does not address: the connection between this treatment of the "elements" and the one discussed in Section 3.1, in which we were told that we had to deal with visible, tangible solid bodies and ended up with an unexplained "proportion":

- Fire is to air as air is to water as water is to earth.

There is no obvious way of correlating this proportion with a true one expressed in terms of triangular pyramid, octahedron, icosahedron, cube. And, although Timaeus does allude to the first proportionality (53e), his failure to make a connection suggests that Plato does not think there is one. Presumably, too, the first proportionality is not intended to apply to the pre-cosmic situation in which only traces of the "elements" exist and are shaken about in the receptacle.

Another issue of connection which is not addressed in detail by Timaeus brings us to a possible difference between the third and fourth interpretations of the epigraph. The third interpretation relates clearly and specifically to the passage I have been discussing. The fourth seems to be an artificial reading of the epigraph which draws an analogy between a particular geometric construction and god's imposition of form on matter, perhaps in order to avoid the specifically geometric interpretation of god's activity, which is now seen as the imposition of form on matter, say, in making a portion of that matter come to have the features of fire, i.e., to resemble the form of fire. It is possible to imagine

[25] Euclid also proves that these are the only five regular solids (XIII.17). In this essay I do not discuss Timaeus' curious remark at 55c about the dodecahedron or the curious discussion of the number of worlds which follows it.

[26] The division of the faces is curious because, Timaeus breaks the square into four isosceles right triangles and the equilateral triangle into six half-equilaterals, although he could have gotten away with two rudimentary triangles in each case. If there is some significance in Timaeus' choice, Plato has not chosen to tell us what it is. For discussion of this question see [19].

[27] Aristotle (*On the Heaven* III.7.306a1–17) ridicules Plato for adopting this position, which he sees as a case of promoting theory over what is observed.

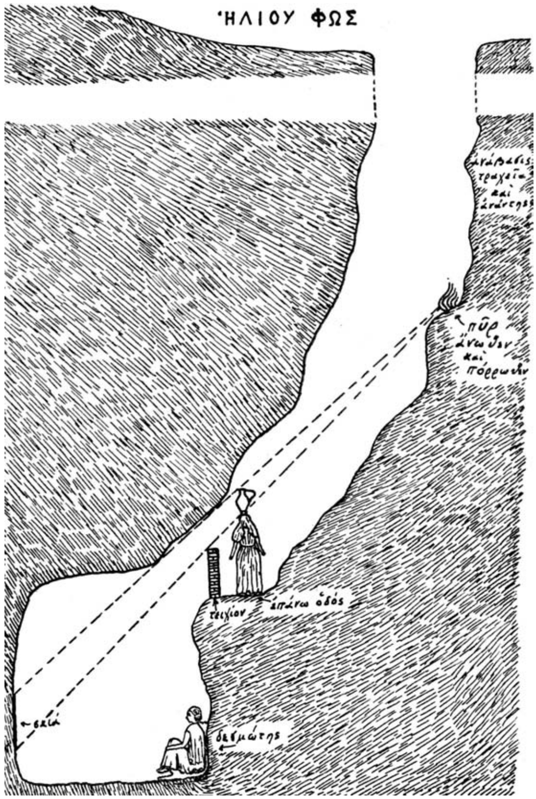

Fig. 3. Plato's cave.[28]

that god imposes the form of fire on matter without going through any geometrical steps. But Plato clearly thinks that God imposes such forms by mathematical means and that these mathematical means "explain" the characteristics of, say, fire. However, Timaeus' explanations of the correlations between the solids and the characteristics are, for us, mere analogies. He begins with earth:

> Let us next distribute the figures whose formation we have described among fire, earth, water, and air. To earth let us assign the cubical figure; for earth is the most immobile of the four kinds and the most moldable of bodies. The figure whose bases are the most stable must best answer that description; and as a base, if we take the ⟨rudimentary⟩ triangles we assumed at the outset, the face of the triangle with equal sides is by nature more stable than that of the triangle whose sides are unequal; and further of the two equilateral surfaces respectively composed of the two triangles, the square is necessarily a more stable base than the triangle, both in its parts and as a whole. Accordingly we shall preserve the plausibility of our account, if we assign this figure to earth. (55d–56a)

The remaining assignments are justified by assuming a correlation between a lower number of faces and greater mobility, sharpness, and lightness. It seems clear that Plato is much

[28] The drawing was made by Dr. B.Th. Koppers. Source: C.J. de Vogel, *Greek Philosophy, A Collection of Texts, Vol. 1, Thales to Plato*, 3rd edition, E.J. Brill, Leiden, 1963, p. 204.

more interested in indicating that there is a mathematical foundation for physics than in giving an account of physical mechanisms by which the foundation is realized.

The question of the relationship between mathematics and physics is made more difficult by the seemingly effortless way with which Timaeus moves between the purely mathematical and the physical. When he begins to describe the order imposed on the disordered receptacle by the demiurge, the four "elements" already exist, and apparently we know they are "bodies". The reduction of bodies to the rudimentary triangles appears to move us from the domain of physics to the domain of pure geometry, but Plato gives no indication of where he thinks the transition occurs. Bodies, he says, must have depth. This is, of course, true of physical bodies, but it is also true of geometrical solids. But as he proceeds to talk about what has depth being bounded by planes and planes being composed of triangles, and so on, he appears to be talking at the geometrical level rather than the physical one. But after he has reverted to the physical level by correlating the solids with the four "elements" and describing their transformations and non-transformations into one another, we are left with an apparently bizarre picture of two-dimensional surfaces floating in space and, where appropriate, combining to form a new solid. It is difficult to suppose that Plato wished to assign a physical reality to these triangles and squares, but equally difficult to see how he could treat them as purely mathematical objects and assign them physical effects.

The difficulties here do not appear to be difficulties for Plato. For him the physical world gains its intelligibility, and therefore, in some sense, its reality, from an ideal world, and, at least in the *Timaeus*, that ideal world is importantly mathematical. But, as I mentioned earlier, the mathematical intelligibility of the physical world is not a matter of what we would call mathematical laws, but rather of a mathematical structure, a system of ratios, a system of interacting spheres or circles, the system of regular solids, being present in the physical world.

3.4. *Mathematics and human happiness*

In the *Timaeus* these mathematical structures are said to be present in the world because a god put them there. And, at least in the case of the heavenly revolutions, our apprehension of them is a way of perfecting ourselves:

> The god invented and gave us vision in order that we might observe the circuits of mind in the heaven and profit by them for the revolution of our own thought, which are of the same kind as them, though ours be troubled and they are unperturbed; and that by learning to know them and acquiring the power to compute them rightly according to nature, we might reproduce the perfectly unerring revolutions of the god and reduce to settled order the wandering motions in ourselves. (47b–c)

This, roughly speaking, moral significance of mathematics is related to the second interpretation of the epigraph, in which geometric proportion is espoused over arithmetic proportion as a basis for social justice. The thought underlying the second interpretation is expressed clearly in Plato's *Laws* at VI.756e–758a, although in terms of two kinds of equality. But the moral significance of mathematics comes out much more clearly in a related passage in the *Gorgias*, where Socrates is in heated exchange with the opportunistic, self-seeking Callicles. Socrates insists that a person should control his appetites:

> He should not allow his appetites to be undisciplined or undertake to fill them up. Such a man, Callicles, could not be loved by another nor by a god, for he cannot participate in a community and without community there is no love. Wise men say, my friend, that community and love and order and moderation and justice hold together heaven and earth, gods and men, and that is why they call this universe a cosmos not a chaos or licentious revel. It seems to me that you do not put your mind to these things, even though you are versed in them; rather, failing to observe the great power of geometric equality among gods and among men, you think you should seek for more and more, because you neglect geometry. (507e–508a)

Putting this passage together with the one just quoted from the *Timaeus*, one can see that Plato brings together mathematical knowledge, morality, and a kind of deification in a way which is hard for a person who is accustomed to the modern compartmentalization of science, ethics, and religiosity to articulate. However we will see the same kind of assimilation in the *Republic*, where, as I have indicated, there is no analog of the god of the *Timaeus*.

3.5. *Higher principles (Timaeus 48b–c and 53d)*

Before turning to the *Republic* I want to signal the two passages in which Timaeus denies that in the second discussion of the "elements" he will present or has presented what we might call absolutely first principles, since these passages have played a role in the discussion of Plato's unwritten doctrines. Timaeus has "reduced" the "elements" to geometric solids and the geometric solids to the rudimentary triangles, but he declines to proceed any further. For those who believe in the unwritten doctrines such further reductions would be included in them. Here I wish to mention only that the most obvious further reduction would be of the triangles to straight lines. Further reductions beyond that, e.g., of straight lines to points, would seem less straightforward although there are indications that Plato thought about some such thing. In addition there are indications that Plato thought about a reduction of geometry to arithmetic and a further reduction of mathematics to something less mathematical or even non-mathematical, namely the forms. All of these ideas are very nebulous and controversial, but I shall have a little more to say about them after discussing the *Republic*.

4. The *Republic*[29]

The central topic of the *Republic* is *dikaiosunê*, a term which applies to both social and political justice and to individual moral rectitude; it is standardly translated 'justice'. However, the work is justifiably often thought of as the fullest representation we have of a

[29] For the *Republic* I have relied on the revised Grube translation [14]. The best annotated edition of the *Republic* remains [1]. Good commentaries include [28] and [3]. The classic essay on much of the material I discuss here is [9].

Platonic philosophy.[30] My concern here is not with Plato's account of justice, but with his depiction of (i) the ideally just person in a ideally just society, the philosopher who will be the ruler of this society, and (ii) his education. For Plato, the distinguishing feature of the philosopher–ruler is his knowledge of the forms and in particular of the form of the good, a knowledge which is displayed dialectically, that is, through successful defense of an account of goodness in the face of sharp and repeated questioning and successful refutation of alternative accounts through the same kind of questioning. The important passages bearing on mathematics occur in books VI and VII of the *Republic* in which the central speaker of the dialogue, Socrates, offers an indirect indication of what the form of the good is and a description of the special education of the philosopher–ruler.

4.1. *The education of the philosopher–ruler*

In books II and III of the *Republic* Socrates describes an elementary education focusing, as Greek education did, on music and literature and physical training. This education is aimed at the development of the moral and physical capacities of the students. It appears that these young people will also be exposed to mathematics by involving them in types of play which develop mathematical facility (VII.536d–e; the idea is developed more fully at VII.819b–d of the *Laws*). At the age of 20 the morally, physically, and intellectually best students will be separated out and begin a higher education with no precedent in fifth-century Greece. For ten years these people are to study mathematics in a serious way, bringing its branches together "into a unified view of their connection with one another and with the nature of reality" (VII.537c). After that the best of these students will spend five years in dialectical questioning and then a select few of them will spend 15 years in governmental and military affairs. Finally the best of these will be brought to an apprehension of the form of the good and spend the remainder of their life alternating between philosophical study and ruling.

Clearly Plato's educational scheme has a moral, a physical, a mathematical, a dialectical, and a practical component. Plato has nothing good to say for people who are highly developed in one of these components at the expense of the others, say, a brilliant mathematician who can't ride a horse or a skilled dialectician with no moral scruples. He sees these components as inextricably blended together in the education of his "divine" (VI.497c) philosopher–rulers. For this reason it is in an important sense misleading to isolate the mathematical component, but this must be done here, and doing it has the merit of bringing to the fore one of Plato's main contributions to western educational theory: the idea that a fully developed human being must be scientifically literate.

The mathematical curriculum

I mentioned at the very beginning of this essay that the five components of mathematical education for Plato are arithmetic, plane and solid geometry, astronomy, and harmonics. In his description of these sciences he does not give a further explanation of how one develops a "unified view of their connection with one another and with the nature of reality", although he does mention the need to do this at 531c–d. His emphasis is almost entirely on

[30] See, for example, [27, p. 115].

what is emphasized in the first interpretation of the epigraph, the way in which mathematics leads us away from the perceptible world to an intelligible reality. Arithmetic is praised because it requires us to deal with units or ones which, unlike single cows or horses, are absolutely indivisible.[31] Socrates wonders what arithmeticians would say if they were asked "What kind of numbers are you talking about, in which the one is as you assume it to be, each equal to every other, without the least difference and without parts", and imagines their answer:

> I think they would answer that they are talking about those which can only be thought about and can't be dealt with in any other way. (VII.526a)

Plane geometry is treated similarly, although Socrates regrets that geometers have to speak as if they were doing things such as constructing squares or moving a figure to a certain position, since the subject matter of geometry is really unchanging reality, not things that come into existence and go out of existence. Socrates does not say anything specific about solid geometry, but only urges that the study of it be promoted.

So, we might say that the first three sciences in Socrates' curriculum are there because they turn the mind away from human things toward divine ones. One might, of course, adopt a position of Aristotle and reject the claim that there are separately existing realities studied by mathematics.[32] But even if one adopts the Platonic view, it is difficult to see how it can be applied to the last two sciences of the curriculum, astronomy and harmonics, which appear to be about things we see and hear. We have seen in Section 3.4 that in the *Timaeus* Plato praises the study of the heavenly motions because by understanding them we see that they are more regular than they appear, and that understanding somehow affects our own emotional psychology. But Socrates goes much further in the *Republic*. The heavens are indeed beautiful, but their observed motions fall far short of their true ones, which must be grasped intellectually not by sight. The true ones are permanent, whereas the observed ones change and do not embody the truth about the "ratio of night to day, of days to a month, of a month to a year, or of the motions of the stars to these things or to each other". Socrates concludes:

> Then if, by really taking part in astronomy, we're to make the naturally intelligent part of the soul useful instead of useless, let us study astronomy by means of problems, as we do geometry, and leave the things in heaven alone. (VII.530b–c)

There are many obscurities in Socrates' discussion of astronomy, but it is clear that Socrates is envisaging a science which is quite different from anything we think of as astronomy, even mathematical astronomy, in which the observed heavenly motions are to some extent idealized to make them more susceptible to geometric representation. I will argue that the astronomical passages of the *Timaeus* are an indication of the sort of thing Plato had in mind for astronomy, but first I want to complete my account of Socrates' mathematical curriculum by mentioning his treatment of harmonics, which is quite parallel to his treatment of astronomy although formulated only negatively. In it Socrates complains about people who "look for numbers in heard concords and do not ascend to problems, investigating

[31] A somewhat different distinction is made at *Philebus* 56d–e between units of different sizes and units which do not differ from one another in any way.
[32] See, for example, Aristotle, *Metaphysics* M.2.1076b11–1077b14.

which numbers are concordant and which aren't and why". (531c) When his interlocutor says that such an investigation is superhuman (*daimonios*), Socrates responds that it is of use in the search for the good and beautiful.

In my description of Timaeus' construction of the world soul in Section 3.1 I pointed out that he partitioned the stuff of the world soul in accordance with the standard diatonic scale without ever mentioning the musical content of his description. And he arranged the stuff into moving circles constituting a rough model of the heavenly motions, but only pointed out that they are a model some two pages after doing the construction. In the *Republic* Plato is concerned with the task of getting the soul aware of a higher level of reality to which the soul is connected. Arithmetic and geometry provide him with examples of sciences which, he thinks, clearly deal with this level. Two other mathematical sciences, astronomy and harmonics, do not do so clearly, but Plato imagines that they could. On the other hand, in the *Timaeus* Plato is concerned to show how our world is derived from the higher level. He does this not only in the two cases I have just mentioned, but also in the explanation of the existence of the four "elements" based on a presumably arithmetic fact about proportion, and of their character on the basis of solid geometry. Timaeus' derivations are not anything like formal deductions, but they are based on the idea that mathematical science is a standard of rationality and that our world, being the best embodiment of a rational organization, is mathematically organized.

4.2. *The divided line (Republic VI.509d–511e)*

The sharp contrast between the perceptible and the intelligible world, which is emphasized in the *Republic* and many other dialogues, is usually treated as a central point of Platonic doctrine. Such a position creates difficulties for the interpretation of the *Timaeus*, which is devoted to an account of the perceptible world. I see this discrepancy between the two dialogues as a matter of emphasis rather than of doctrine, a position which I would like to buttress by looking at one further passage, part of Socrates' attempt to indicate in a rough way the nature of the form of the good.

The first part of that attempt is a comparison of the relationship of the sun to the perceptible world to that of the form of the good to the intelligible world. The second part, usually referred to as The Divided Line, compares the distinction between the intelligible and the sensible world to the division of a straight line AB into two unequal sections at C. However, Socrates then imagines a further division of AC at A_1 and CB at B_1 so that:

$$AC : CB :: AA_1 : A_1C :: CB_1 : B_1B.$$

The difference between CB_1 and B_1B is explained in terms of the distinction between perceptible things, animals, plants, artifacts (CB_1), and their images, shadows, reflections, etc. (B_1B). But the difference between AA_1 and A_1C is made in terms of a distinction between two intellectual methods. A_1C is explained by reference to geometers, arithmeticians, and the like, who hypothesize various things as known and give no account of them and then make inferences based on them; these people are also said to make use of visible

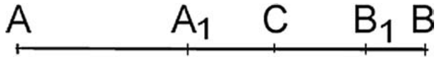

Fig. 4.

things in their reasoning even though they are not thinking about these things but about the things of which they are likenesses. The best model we have for this is probably elementary geometry done in the style of Euclid, which starts from definitions, postulates, and axioms, and reasons in terms of diagrams. AA_1 is associated with dialectic, which is said to use its hypotheses as underpinnings for rising to an "unhypothetical principle of everything" (certainly the form of the good) and then to move down to a conclusion, not using sensible things, but only dealing with forms.

In the context of the material we have been discussing it seems natural to think that A_1C is exhausted by mathematics, that is, presumably, the five subjects Socrates puts into his mathematical curriculum. At *Metaphysics* A.5.987b14–18 Aristotle asserts that Plato believed in mathematical objects which were like the forms and unlike perceptibles in being eternal and like sensibles and unlike forms in coming in multiple instances. Such a distinction might make some sense for arithmetic and geometry, but, as Aristotle himself points out (*Metaphysics* M.2.1076b39–1077a9 and B.2.997b12–24), it makes little sense for astronomy and harmonics. However, in the *Republic* nothing is said about a distinction between the objects in AC and A_1C, and, indeed, later (VII.534a) Socrates explicitly declines to enter into this issue. A striking feature of the way Socrates divides the line, but one which is not mentioned in the *Republic* or elsewhere in antiquity, is that it makes A_1C and CB_1 equal. I am not entirely convinced that Plato was aware of this fact, but the equality and Socrates' refusal to discuss ontological issues here might well be a way of expressing the nebulousness of the domain of mathematics, its transitional role between the sensible and the intelligible, both in the upward direction in the *Republic* and in the downward direction in the *Timaeus*.

5. Conclusion

To many modern readers Socrates' presentation of the divided line suggests the idea of a deductive system which in some mysterious way starts from the form of the good as an unhypothetical first principle and deduces an exhaustive description of the forms and the body of mathematical knowledge. My own sense is that the "downward movement" referred to in the connection with AA_1 and A_1C includes the kinds of applications of mathematics we see in the *Timaeus* and other kinds of reasoning which we would not class as deductive. Plato, of course, never provides anything appreciably more detailed than the kind of schematic presentation I have outlined here. For proponents of the unwritten doctrines, the absence of anything more detailed is only to be expected. The dispute over the unwritten doctrines is importantly a dispute over their significance for Plato. For me that question remains open, although I am inclined to think that some people have exaggerated their significance by making them the key to understanding all of Plato. Putting that issue aside, and assuming, what has also been disputed, that Plato did have the idea of some broad doctrine of the kind I have tried to sketch, there would still remain the question of how detailed and specific that doctrine was. I think that question is unanswerable, although it is important to say that we have no information about a detailed and specific doctrine.

However, even without that information, it seems clear to me that Plato did have a comprehensive picture of the cosmos divided into a higher divine, intelligible world and a lower human, sensible world, and saw the task of the individual, on which his well-being, his divinization, depends as a matter of somehow attaching himself to that higher world. A, and I suspect the, crucial link between these worlds is mathematics, in the broad sense determined by the mathematical curriculum of the *Republic*. In the *Republic* Plato emphasizes the role of mathematics in directing the attention of the potentially divine individual away from the lower world to the higher world and its apex, the form of the good. But in the *Timaeus* he emphasizes the way in which the higher world is expressed in the mathematical organization of the lower world. In both dialogues and elsewhere Plato insists on the moral benefit to be gained from the study of mathematics.

These fundamental beliefs of Plato found their expression in what has been called the world's first institution of higher learning, Plato's Academy. Having organized this essay around one, probably apocryphal, statement ascribed to Plato, I want to end with another, the inscription Plato is said to have put over the entrance to his Academy:

Let no one unversed in geometry enter here![33]

Appendix A. The division of the stuff of the world soul (*Timaeus* 35b–36b)

Here is Timaeus' description of the way in which the demiurge divided up the stuff of the world soul before reassembling it and forming it into circles corresponding to major heavenly motions:

> And he began the division in this way.
> (a) First, he took one portion from the whole, and next a portion double this; the third half as much again as the second, and three times the first; the fourth double the second; the fifth three times the third; the sixth eight times the first; and the seventh twenty-seven times the first.
> (b) Next he went on to fill up both the double and triple intervals, cutting off yet more parts from the original mixture and placing them between the terms, so that within each interval there were two means, (b1) the one exceeding and being exceeded by the extremes by the same part, (b2) the other exceeding and being exceeded by an equal number. These links gave rise to intervals of three to two and four to three and nine to eight within the original intervals.
> (c) And he went on to fill up all the intervals of four to three with the interval of nine to eight, leaving a part over in each of them. This leftover interval of the part had the ratio of the number two hundred and fifty six to the number two hundred and forty three.

The demiurge's procedure is the following.

(a) He first generates the series of powers of 2 and 3 in the order:

1 2 3 4 9 8 27

[33]"*Ageômetrêtos mêdeis eisetô*". On this inscription see [23].

(b) He then imagines these arranged as intervals (or ratios):

1	2	1	3
2	4	3	9
4	8	9	27

and fills these intervals $m \quad n$ with numbers k, l such that (b1) $k - m$ is to m as $n - k$ is to n and (b2) $l = \frac{m+n}{2}$, the so-called harmonic and arithmetic means between m and n. This yields:

1	4/3	3/2	2	1	3/2	2	3
2	8/3	3	4	3	9/2	6	9
4	16/3	6	8	9	27/2	18	27

Combining these two sequences we have:

1	4/3	4	3
4/3	3/2	9	8
3/2	2	4	3
2	8/3	4	3
8/3	3	4	3
3	4	4	3
4	9/2	9	8
9/2	16/3	32	27
16/3	6	9	8
6	8	4	3
8	9	9	8
9	27/2	3	2
27/2	18	4	3
18	27	3	2

where the right column expresses the ratio of the limiting terms in least numbers. Timaeus says that this process produces intervals representing the ratios 3/2, 4/3, and 9/8, which is true except for the case of 9/2   6/3.

(c) Timaeus then says the demiurge filled up the 4   3 intervals with 9   8 intervals leaving an interval of 256   243. These values correspond to the fact that $4/3 = 9/8 \cdot 9/8 \cdot 256/243$. Represented in terms of intervals one would have:

1   9/8   81/64   4/3

and these numbers would correspond to the first four notes of the diatonic scale, which I shall call do, re, mi, fa. Continuing we move up through two octaves as follows:

1			4/3	4	3
do	re	mi	fa		
4/3			3/2	9	8
fa			sol		
3/2			2	4	3
sol	la	ti	do′		
2			8/3	4	3
do′	re′	mi′	fa′		
8/3			3	9	8
fa′			sol′		
3			4	4	3
sol′	la′	ti′	do′′		

The continuation is a little less straightforward, since for the next octave we start from:

4			9/2	9	8
do′′			re′′		
9/2			16/3	32	27
re′′			fa′′		
16/3			6	9	8
fa′′			sol′′		
6			8	4	3
sol′′	la′′	ti′′	do′′′		

Although he does not say so, Timaeus presumably intends us to consider the combined interval:

4	16/3	4	3

and fill it up as in the case of the preceding two octaves. The remaining octave plus is represented by:

8	9	9	8
do′′′	re′′′		
9	27/2	3	2
re′′′	la′′′		

27/2 la$'''$	18 re$''''$	4	3
18 re$''''$	27 la$''''$	3	2

la$'''$ and re$'''''$ give us another 4 3 interval, which we can fill up in accordance with the diatonic scale if we use the order:

$$1 \quad 9/8 \quad 2304/1944 \quad 4/3$$

but there does not seem to be any way that the demiurge can construct quantities for mi$'''$, fa$'''$, sol$'''$ or mi$''''$, fa$''''$, sol$''''$ following Timaeus' recipe.

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