Logarithms on Time Scales

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Open Problem

Define a "nice" logarithm function on time scales and present its properties.

Previous Logarithms on Time Scales

- S. Huff, G. Olumolode, N. Pennington, and A. Peterson:

$$\int_{t_0}^{t} \frac{2}{\tau + \sigma(\tau)} \Delta\tau$$

- M. Bohner: $\ln t$ analogous to

$$\int_{t_0}^{t} \frac{1}{\tau + 2\mu(\tau)} \Delta\tau$$

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- Bohner: for Δ-differentiable $p : \mathbb{T} \to \mathbb{R}$,

$$L_p(t, t_0) = \int_{t_0}^{t} \frac{p^\Delta(\tau)}{p(\tau)} \Delta\tau$$

- B. Jackson: for Δ-differentiable $p : \mathbb{T} \to \mathbb{R}$,

$$\log_{\mathbb{T}} p(t) = \frac{p^\Delta(t)}{p(t)}$$

- Mozyrska & Torres: for time scales such that $1 \in \mathbb{T}$,

$$L_{\mathbb{T}}(t) = \int_{1}^{t} \frac{1}{\tau} \Delta\tau$$

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Single-Valued Cylinder Transformation

Definition (Hilger, Bohner & Peterson)

For $h > 0$, define the cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ by

$$\xi_h(z) = \begin{cases} \frac{1}{h}\operatorname{Log}(1+zh) & \text{for } h \neq 0 \\ z & \text{for } h = 0, \end{cases} \tag{1}$$

where $\mathbb{C}$ is the set of complex numbers,

$$\mathbb{C}_h = \left\{ z \in \mathbb{C} : \ z \neq -\frac{1}{h} \right\}, \quad \mathbb{Z}_h = \left\{ z \in \mathbb{C} : \ -\frac{\pi}{h} < \operatorname{Im}(z) \leq \frac{\pi}{h} \right\},$$

and Log is the principal logarithm function.

Regressive Function

Definition

A function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}^\kappa$$

holds. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$.

Exponential Function

Definition

For functions $p \in \mathcal{R}$, the exponential function on time scales is given by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right) \quad \text{for} \quad s,t \in \mathbb{T},$$

where $\xi_h(z)$ is the cylinder transformation.

We now set the foundation for offering a new definition of logarithms on time scales. This definition will be of a multi-valued function, for which we need to modify the single-valued cylinder function.

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We now set the foundation for offering a new definition of logarithms on time scales. This definition will be of a multi-valued function, for which we need to modify the single-valued cylinder function.

Definition

For $h > 0$, define the multi-valued cylinder transformation $\zeta_h : \mathbb{C}_h \to \mathbb{C}$ by

$$\zeta_h(z) = \begin{cases} \dfrac{1}{h}\log(1 + zh) & \text{for } h \neq 0 \\ z & \text{for } h = 0, \end{cases} \tag{2}$$

where $\mathbb{C}$ is the set of complex numbers and log is the multi-valued complex logarithm function.

Multi-Cylinder Addition Property

Lemma

Let $f, g : \mathbb{T} \to \mathbb{C}$ be Δ-differentiable functions with $f, g \neq 0$ on $\mathbb{T}$, and let the multi-valued cylinder transformation ζ be given by (2). Then, for fixed $\tau \in \mathbb{T}^{\kappa}$,

$$\zeta_{\mu(\tau)}\left(\left(\frac{f^{\Delta}}{f} \oplus \frac{g^{\Delta}}{g}\right)(\tau)\right) = \zeta_{\mu(\tau)}\left(\frac{f^{\Delta}(\tau)}{f(\tau)}\right) + \zeta_{\mu(\tau)}\left(\frac{g^{\Delta}(\tau)}{g(\tau)}\right).$$

Proof of Addition Property: Key Idea

First, note that the simple useful formula $f^\sigma = f + \mu f^\Delta$ (suppressing the variable) implies

$$\begin{aligned}
\frac{(fg)^\Delta}{fg} &= \frac{f^\sigma g^\Delta + f^\Delta g}{fg} \\
&= \frac{(f + \mu f^\Delta) g^\Delta}{fg} + \frac{f^\Delta}{f} \\
&= \frac{f^\Delta}{f} + \frac{g^\Delta}{g} + \mu \frac{f^\Delta g^\Delta}{fg} \\
&= \frac{f^\Delta}{f} \oplus \frac{g^\Delta}{g}.
\end{aligned}$$

It follows that for fixed $\tau \in \mathbb{T}^\kappa$,

$$\zeta_{\mu(\tau)}\left(\left(\frac{f^\Delta}{f} \oplus \frac{g^\Delta}{g}\right)(\tau)\right) = \zeta_{\mu(\tau)}\left(\frac{(fg)^\Delta(\tau)}{(fg)(\tau)}\right)$$

$$= \begin{cases} \frac{1}{\mu(\tau)}\log\left(1+\mu(\tau)\frac{(fg)^\Delta(\tau)}{(fg)(\tau)}\right) & \text{for } \mu(\tau)\neq 0 \\ \frac{(fg)^\Delta(\tau)}{(fg)(\tau)} & \text{for } \mu(\tau)=0 \end{cases}$$

$$= \begin{cases} \frac{1}{\mu(\tau)}\log\left(\frac{(fg)^\sigma(\tau)}{(fg)(\tau)}\right) & \text{for } \mu(\tau)\neq 0 \\ \left(\frac{f^\Delta}{f} \oplus \frac{g^\Delta}{g}\right)(\tau) & \text{for } \mu(\tau)=0 \end{cases}$$

$$= \begin{cases} \frac{1}{\mu(\tau)}\log\left(\frac{(f+\mu f^\Delta)(\tau)}{f(\tau)}\right) + \frac{1}{\mu(\tau)}\log\left(\frac{(g+\mu g^\Delta)(\tau)}{g(\tau)}\right) \\ \frac{f^\Delta(\tau)}{f(\tau)} + \frac{g^\Delta(\tau)}{g(\tau)} \end{cases}$$

$$= \zeta_{\mu(\tau)}\left(\frac{f^\Delta(\tau)}{f(\tau)}\right) + \zeta_{\mu(\tau)}\left(\frac{g^\Delta(\tau)}{g(\tau)}\right).$$

This completes the proof.

Multi-Cylinder Exponent Property

Lemma

Let $\alpha \in \mathbb{R}$, *and let* $p : \mathbb{T} \to \mathbb{C}$ *be a* Δ*-differentiable function with* $p \neq 0$ *on* $\mathbb{T}$*. For the multi-valued cylinder transformation* ζ *given by* (2) *and for fixed* $\tau \in \mathbb{T}^\kappa$,

$$\zeta_{\mu(\tau)}\left(\left(\alpha \odot \frac{p^\Delta}{p}\right)(\tau)\right) = \alpha\zeta_{\mu(\tau)}\left(\frac{p^\Delta(\tau)}{p(\tau)}\right).$$

Proof of the Multi-Cylinder Exponent Property

$$
\begin{aligned}
&\zeta_{\mu(\tau)}\left(\left(\alpha \odot \frac{p^\Delta}{p}\right)(\tau)\right) \\
&= \begin{cases} \dfrac{1}{\mu(\tau)} \log\left(1+\mu(\tau)\left(\alpha \odot \dfrac{p^\Delta}{p}\right)(\tau)\right) & \text{for } \mu(\tau) \neq 0 \\ \left(\alpha \odot \frac{p^\Delta}{p}\right)(\tau) & \text{for } \mu(\tau) = 0 \end{cases} \\
&= \begin{cases} \dfrac{1}{\mu(\tau)} \log\left(1+\mu(\tau)\dfrac{p^\Delta(\tau)}{p(\tau)}\right)^\alpha & \text{for } \mu(\tau) \neq 0 \\ \alpha\frac{p^\Delta(\tau)}{p(\tau)} & \text{for } \mu(\tau) = 0 \end{cases} \\
&= \alpha \begin{cases} \dfrac{1}{\mu(\tau)} \log\left(1+\mu(\tau)\dfrac{p^\Delta(\tau)}{p(\tau)}\right) & \text{for } \mu(\tau) \neq 0 \\ \frac{p^\Delta(\tau)}{p(\tau)} & \text{for } \mu(\tau) = 0 \end{cases} \\
&= \alpha\zeta_{\mu(\tau)}\left(\frac{p^\Delta(\tau)}{p(\tau)}\right). \qquad \square
\end{aligned}
$$

New Logarithm Function

Definition

For a Δ-differentiable function $p : \mathbb{T} \to \mathbb{C}$ with $p \neq 0$ on $\mathbb{T}$, the multi-valued logarithm function on time scales is given by

$$\ell_p(t,s) = \int_s^t \zeta_{\mu(\tau)}\left(\frac{p^\Delta(\tau)}{p(\tau)}\right)\Delta\tau \quad \text{for} \quad s,t \in \mathbb{T},$$

where $\zeta_h(z)$ is the multi-valued cylinder transformation given in (2). Define the principal logarithm on time scales to be

$$L_p(t,s) = \int_s^t \xi_{\mu(\tau)}\left(\frac{p^\Delta(\tau)}{p(\tau)}\right)\Delta\tau \quad \text{for} \quad s,t \in \mathbb{T},$$

where $\xi_h(z)$ is the single-valued cylinder transformation given in (1).

Remark

According to this definition, if $p \equiv$ *constant, then* $\ell_p(t,s) = 0$ *for all* $t, s \in \mathbb{T}$. *Thus, this logarithm distinguishes between neither constants nor constant multiples of functions.*

For $\mathbb{T} = \mathbb{R}$,

$$\ell_p(t,s) = \int_s^t \zeta_{\mu(\tau)}\left(\frac{p^\Delta(\tau)}{p(\tau)}\right)\Delta\tau = \int_s^t \frac{p'(\tau)}{p(\tau)}\mathrm{d}\tau = \log\left(\frac{p(t)}{p(s)}\right),$$

where log is the multi-valued complex logarithm function.

$\mathbb{T} = h\mathbb{Z}$: $f^\Delta(\tau) = \Delta_h f(\tau) = \frac{f(\tau+h)-f(\tau)}{h}$

$$
\begin{aligned}
\ell_p(t,s) &= \int_s^t \zeta_{\mu(\tau)}\left(\frac{p^\Delta(\tau)}{p(\tau)}\right)\Delta\tau \\
&= \sum_{j=\frac{s}{h}}^{\frac{t}{h}-1} \zeta_h\left(\frac{\Delta_h p(jh)}{p(jh)}\right) h \\
&= \sum_{j=\frac{s}{h}}^{\frac{t}{h}-1} \frac{1}{h}\log\left(1+\frac{h\Delta_h p(jh)}{p(jh)}\right) h \\
&= \sum_{j=\frac{s}{h}}^{\frac{t}{h}-1} \log\left(\frac{p(jh+h)}{p(jh)}\right) = \log\left(\prod_{j=\frac{s}{h}}^{\frac{t}{h}-1}\frac{p((j+1)h)}{p(jh)}\right) \\
&= \log\left(\frac{p(t)}{p(s)}\right).
\end{aligned}
$$

$\mathbb{T} = q^{\mathbb{N}_0}$, $q-$difference equations

$$p^{\Delta}(\tau) := \frac{p(q\tau) - p(\tau)}{(q-1)\tau} \implies$$

$$\begin{aligned}
\ell_p(t,s) &= \int_s^t \zeta_{\mu(\tau)}\left(\frac{p^{\Delta}(\tau)}{p(\tau)}\right)\Delta\tau \\
&= \sum_{\tau\in[s,t)} \zeta_{(q-1)\tau}\left(\frac{p^{\Delta}(\tau)}{p(\tau)}\right)(q-1)\tau \\
&= \sum_{\tau\in[s,t)} \frac{1}{(q-1)\tau}\log\left(1+\frac{(q-1)\tau p^{\Delta}(\tau)}{p(\tau)}\right)(q-1)\tau \\
&= \sum_{\tau\in[s,t)} \log\left(\frac{p(q\tau)}{p(\tau)}\right) \\
&= \log\left(\frac{p(t)}{p(s)}\right).
\end{aligned}$$

$\mathbb{T} = [a, b] \cup [c, d]$

Assume without loss of generality that $s \in [a, b]$ and $t \in [c, d]$. Then $c = \sigma(b)$, and

$$\begin{aligned}
\ell_p(t, s) &= \int_s^t \zeta_{\mu(\tau)} \left(\frac{p^\Delta(\tau)}{p(\tau)} \right) \Delta\tau \\
&= \left(\int_s^b + \int_b^{\sigma(b)} + \int_{\sigma(b)}^t \right) \zeta_{\mu(\tau)} \left(\frac{p^\Delta(\tau)}{p(\tau)} \right) \Delta\tau \\
&= \log\left(\frac{p(b)}{p(s)} \right) + \log\left(\frac{p(t)}{p(c)} \right) + \mu(b)\zeta_{\mu(b)} \left(\frac{p^\Delta(b)}{p(b)} \right) \\
&= \log\left(\frac{p(b)}{p(s)} \right) + \log\left(\frac{p(t)}{p(c)} \right) + \log\left(\frac{p^\sigma(b)}{p(b)} \right) \\
&= \log\left(\frac{p(b)}{p(s)} \right) + \log\left(\frac{p(t)}{p(c)} \right) + \log\left(\frac{p(c)}{p(b)} \right) \\
&= \log\left(\frac{p(t)}{p(s)} \right).
\end{aligned}$$

Example: Principal Logarithm

Example

Let $\mathbb{T} = (-\infty, -4] \cup [2, \infty)$, *and* $p(t) = t^3$. *Let* $t = 3$ *and* $s = -5$. *Then*

$$\mu(-4) = \sigma(-4) - (-4) = 2 - (-4) = 6,$$

and the principal logarithm on this time scale is

$$\begin{aligned} L_p(t,s) &= L_p(3,-5) \\ &= \int_{-5}^{3} \xi_{\mu(\tau)}\left(\frac{(\tau^3)^\Delta}{\tau^3}\right)\Delta\tau \\ &= \left(\int_{-5}^{-4} + \int_{-4}^{2} + \int_{2}^{3}\right)\xi_{\mu(\tau)}\left(\frac{\sigma(\tau)^2 + \tau\sigma(\tau) + \tau^2}{\tau^3}\right)\Delta\tau \\ &= \mathrm{Log}\left(\frac{27}{-125}\right) = \ln\left(\frac{27}{125}\right) + i\pi. \end{aligned}$$

Inverse Property of the Logarithm

Theorem

Let $p : \mathbb{T} \to \mathbb{C}$ be a Δ-differentiable function with $p \neq 0$ on $\mathbb{T}$. Then, for $s, t \in \mathbb{T}$, we have

$$\exp\left(L_p(t,s)\right) = e_{\frac{p^\Delta}{p}}(t,s).$$

Corollary

Let $p \in \mathcal{R}$ and $s, t \in \mathbb{T}$. Then

$$\exp\left(L_{e_p}(t,s)\right) = e_p(t,s).$$

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Let $p \in \mathcal{R}$ and $s, t \in \mathbb{T}$. Then

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Product & Quotient Properties of the Logarithm

Theorem

Let $f, g : \mathbb{T} \to \mathbb{C}$ be Δ-differentiable functions with $f, g \neq 0$ on $\mathbb{T}$. Then, for $s, t \in \mathbb{T}$, we have

$$\ell_{fg}(t,s) = \ell_f(t,s) + \ell_g(t,s)$$

and

$$\ell_{\frac{f}{g}}(t,s) = \ell_f(t,s) - \ell_g(t,s).$$

Proof.

Let $f, g : \mathbb{T} \to \mathbb{R}$ be Δ-differentiable functions with $f, g \neq 0$ on $\mathbb{T}$. Then, for $s, t \in \mathbb{T}$, we have via Lemma 5 and its proof that

$$\begin{aligned}
\ell_{fg}(t,s) &= \int_s^t \zeta_{\mu(\tau)}\left(\frac{(fg)^\Delta(\tau)}{(fg)(\tau)}\right)\Delta\tau \\
&= \int_s^t \zeta_{\mu(\tau)}\left(\left(\frac{f^\Delta}{f} \oplus \frac{g^\Delta}{g}\right)(\tau)\right)\Delta\tau \\
&= \int_s^t \zeta_\mu\left(\frac{f^\Delta}{f}\right)(\tau)\Delta\tau + \int_s^t \zeta_\mu\left(\frac{g^\Delta}{g}\right)(\tau)\Delta\tau \\
&= \ell_f(t,s) + \ell_g(t,s).
\end{aligned}$$

In a similar manner,

$$\ell_{\frac{f}{g}}(t,s) = \ell_f(t,s) - \ell_g(t,s).$$

□

Theorem

Let $\alpha \in \mathbb{R}$, and let $p : \mathbb{T} \to \mathbb{C}$ be a Δ-differentiable function with $p \neq 0$ on $\mathbb{T}$. Then, for $s, t \in \mathbb{T}$, we have

$$\ell_{p^\alpha}(t, s) = \alpha \ell_p(t, s).$$

Proof.

Key ideas:

$$\frac{(p^\alpha)^\Delta}{p^\alpha} = \alpha \odot \frac{p^\Delta}{p},$$

$$\zeta_{\mu(\tau)}\left(\left(\alpha \odot \frac{p^\Delta}{p}\right)(\tau)\right) = \alpha \zeta_{\mu(\tau)}\left(\frac{p^\Delta(\tau)}{p(\tau)}\right).$$

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Theorem

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$$\zeta_{\mu(\tau)}\left(\left(\alpha \odot \frac{p^\Delta}{p}\right)(\tau)\right) = \alpha \zeta_{\mu(\tau)}\left(\frac{p^\Delta(\tau)}{p(\tau)}\right).$$

□

Derivative of Logarithms

Theorem

Let $p : \mathbb{T} \to \mathbb{R}$ be a Δ-differentiable function with $p \neq 0$ on $\mathbb{T}$. Then, for $s, t \in \mathbb{T}$, we have

$$\ell_p^{\Delta}(t,s) = \begin{cases} \frac{1}{\mu(t)} \log\left(\frac{p^{\sigma}(t)}{p(t)}\right) & \text{for } \mu(t) \neq 0 \\ \frac{p^{\Delta}(t)}{p(t)} & \text{for } \mu(t) = 0, \end{cases}$$

where Δ-differentiation is with respect to t.

Proof.

Key idea:

$$\ell_p^{\Delta}(t,s) = \zeta_{\mu(t)}\left(\frac{p^{\Delta}(t)}{p(t)}\right).$$

□

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Proof.

Key idea:

$$\ell_p^{\Delta}(t,s) \quad = \quad \zeta_{\mu(t)}\left(\frac{p^{\Delta}(t)}{p(t)}\right).$$

□

Example

Let $t \in \mathbb{T}$ *with* $t \neq 0$*, and set* $p(t) = t$*. For* $s \in \mathbb{T}$*, we have*

$$\ell_p^{\Delta}(t,s) = \begin{cases} \dfrac{1}{\mu(t)} \log\left(\dfrac{\sigma(t)}{t}\right) & \text{for } \mu(t) \neq 0 \\ \dfrac{1}{t} & \text{for } \mu(t) = 0, \end{cases}$$

where Δ*-differentiation is with respect to* t*. Thus,*

$$\ell_p^{\Delta}(t,s) = \begin{cases} \dfrac{1}{t} & \text{for } \mathbb{T} = \mathbb{R} \\ \dfrac{1}{h} \log\left(1 + \dfrac{h}{t}\right) & \text{for } \mathbb{T} = h\mathbb{Z} \\ \dfrac{\log(q)}{(q-1)t} & \text{for } \mathbb{T} = q^{\mathbb{N}_0}, \end{cases}$$

where $h > 0$ *and* $q > 1$*.*

Example

Consider a discrete time scale with alternating graininess function. In particular, for the two step sizes $\alpha, \beta > 0$ *with* $\alpha \neq \beta$, *let*

$$\mathbb{T}_{\alpha,\beta} = \{0, \alpha, (\alpha+\beta), (\alpha+\beta)+\alpha, 2(\alpha+\beta), 2(\alpha+\beta)+\alpha, \cdots\}.$$

Then, for $t \in \mathbb{T}$ *and* $k \in \mathbb{N}_0 = \{0, 1, 2, 3, 4, \cdots\}$, *we have*

$$\mu(t) = \begin{cases} \alpha & \text{for } t = k(\alpha+\beta), \\ \beta & \text{for } t = k(\alpha+\beta)+\alpha. \end{cases}$$

Set $p(t) = t$. *We claim that for* $t \in \mathbb{T}_{\alpha,\beta}$ *with* $t \neq 0$,

$$\ell_p^{\Delta}(t,s) = \begin{cases} \frac{1}{\alpha}\log\left(1+\frac{\alpha}{t}\right) & \text{for } t = k(\alpha+\beta) \\ \frac{1}{\beta}\log\left(1+\frac{\beta}{t}\right) & \text{for } t = k(\alpha+\beta)+\alpha. \end{cases}$$

Cieśliński introduced an improved exponential function (or the Cayley-exponential function) on a time scale defined by

$$E_p(t,s) = \exp\left(\int_s^t \Psi_{\mu(\tau)}(p(\tau))\Delta\tau\right), \tag{3}$$

where $p : \mathbb{T} \to \mathbb{C}$ is rd-continuous and satisfies the regressivity condition $\mu(\tau)p(\tau) \neq \pm 2$ for all $\tau \in \mathbb{T}^\kappa$, and the modified cylinder transformation Ψ is given by

$$\Psi_h(z) = \frac{1}{h}\,\mathrm{Log}\left(\frac{1+\frac{1}{2}zh}{1-\frac{1}{2}zh}\right), \quad \Psi_0(z) = z, \tag{4}$$

for $h > 0$. Here again, Log represents the principal complex logarithm. Consider the multi-valued function version of (4) denoted, i.e.,

$$\psi_h(z) = \frac{1}{h}\log\left(\frac{1+\frac{1}{2}zh}{1-\frac{1}{2}zh}\right), \quad \psi_0(z) = z. \tag{5}$$

Definition

For a Δ-differentiable function $p : \mathbb{T} \to \mathbb{C}$ with $p \neq 0$ on $\mathbb{T}$, the multi-valued Cayley-logarithm function on time scales is given by

$$\mathrm{caylog}_p(t,s) = \int_s^t \psi_{\mu(\tau)}\left(\frac{2p^\Delta(\tau)}{p(\tau)+p^\sigma(\tau)}\right)\Delta\tau \quad \text{for} \quad s,t \in \mathbb{T},$$

where $\psi_h(z)$ is the multi-valued cylinder transformation given in (5). Define the principal Cayley-logarithm on time scales to be

$$\mathrm{CayLog}_p(t,s) = \int_s^t \Psi_{\mu(\tau)}\left(\frac{2p^\Delta(\tau)}{p(\tau)+p^\sigma(\tau)}\right)\Delta\tau \quad \text{for} \quad s,t \in \mathbb{T},$$

where $\Psi_h(z)$ is the single-valued cylinder transformation given in (4).

Lemma

The Cayley-logarithm functions are well-defined functions.

Proof.

Show that

$$\mu(\tau)\frac{2p^{\Delta}(\tau)}{p(\tau)+p^{\sigma}(\tau)} \neq \pm 2$$

(regressivity condition). The following are equivalent:

$$\begin{aligned}
\frac{2\mu(\tau)p^{\Delta}(\tau)}{p(\tau)+p^{\sigma}(\tau)} &= \pm 2 \\
\frac{p^{\sigma}(\tau)-p(\tau)}{p(\tau)+p^{\sigma}(\tau)} &= \pm 1 \\
p^{\sigma}(\tau)-p(\tau) &= \pm\left(p(\tau)+p^{\sigma}(\tau)\right) \\
p^{\sigma}(\tau)\mp p^{\sigma}(\tau) &= p(\tau)\pm p(\tau),
\end{aligned}$$

so that we have either $0 = 2p(\tau)$ or $2p^{\sigma}(\tau) = 0$, both contradictions. □

Cayley-logarithm functions are equal to earlier logs

Theorem

For a Δ-differentiable function $p : \mathbb{T} \to \mathbb{C}$ with $p \neq 0$ on $\mathbb{T}$,

$$\text{caylog}_p(t,s) = \ell_p(t,s) \quad \textit{and} \quad \text{CayLog}_p(t,s) = L_p(t,s) \qquad (6)$$

for all $t, s \in \mathbb{T}$.

Remark (Generalization)

Let $\eta \in [0,1]$, *and set*

$$\psi_h^\eta(z) = \frac{1}{h}\log\left(\frac{1+(1-\eta)hz}{1-\eta hz}\right), \quad \psi_0^\eta(z) = z. \tag{7}$$

For a Δ*-differentiable function* $p : \mathbb{T} \to \mathbb{C}$ *with* $p \neq 0$ *on* $\mathbb{T}$,

$$\begin{aligned} &\psi_{\mu(\tau)}^\eta\left(\frac{p^\Delta(\tau)}{(1-\eta)p(\tau)+\eta p^\sigma(\tau)}\right) \\ &= \frac{1}{\mu(\tau)}\log\left(\frac{1+(1-\eta)\mu(\tau)\frac{p^\Delta(\tau)}{(1-\eta)p(\tau)+\eta p^\sigma(\tau)}}{1-\eta\mu(\tau)\frac{p^\Delta(\tau)}{(1-\eta)p(\tau)+\eta p^\sigma(\tau)}}\right) \\ &= \zeta_{\mu(\tau)}\left(\frac{p^\Delta(\tau)}{p(\tau)}\right). \end{aligned}$$

Therefore, $\log_p^\eta(t,s) = \ell_p(t,s)$.

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Domo Arigato Gozaimasu!