Isolated Time Scales, Periodicity, and Stability

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Abstract

Recently, a new definition of periodicity for functions defined on isolated time scales has been introduced. We will illustrate this for $q-$difference equations, and then use these periodic functions in simple first-order difference equations to explore questions of Hyers-Ulam stability.

$q-$Difference Equations

For $q > 1$ and the time scale domain $\mathbb{T} = \{1, q, q^2, q^3, \ldots\}$, define the derivative to be

$$z^{\Delta}(t) = \frac{z(qt) - z(t)}{qt - t}, \qquad t \in \mathbb{T}.$$

Traditionally, a periodic function of period ω satisfies $p(t + \omega) = p(t)$ for all t in the domain of p. A new definition by Bohner and Chieochan is

$$q^{\omega} p(q^{\omega} t) = p(t), \qquad t \in \mathbb{T}.$$

Area Interpretation

If p is $\omega-$periodic, then

$$\int_{t}^{q^{\omega}t} p(s)\Delta s = \int_{1}^{q^{\omega}} p(s)\Delta s.$$

So, for $q = 2$, $p(t) = \frac{c}{t}$ is $1-$periodic, as

$$2p(2t) = \frac{2c}{2t} = \frac{c}{t} = p(t).$$

Geometric View for $q = 2$

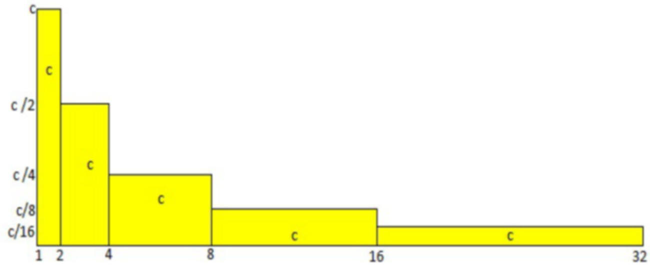

FIGURE 3.1. The constant area of the rectangles corresponding to the 1-periodic function f on the intervals $[2^n, 2^{n+1}]$, $n \in \{0, 1, 2, 3, 4\}$.

Isolated Time Scales

For a general isolated time scale $\mathbb{T}$, let $\sigma(t)$ represent the successor of t in $\mathbb{T}$, and define the derivative to be

$$z^{\Delta}(t) = \frac{z(\sigma(t)) - z(t)}{\sigma(t) - t}, \qquad t \in \mathbb{T}.$$

For example, for $q > 1$ and the time scale $\mathbb{T} = \{1, q, q^2, q^3, \ldots\}$, we saw that

$$z^{\Delta}(t) = \frac{z(qt) - z(t)}{qt - t}, \qquad t \in \mathbb{T}.$$

Periodicity for Isolated Time Scales

For a general isolated time scale $\mathbb{T}$, Bohner, Mesquita, and Streipert (2022) say a function $p : \mathbb{T} \to \mathbb{R}$ is $\omega-$periodic provided

$$(\sigma^{\omega})^{\Delta}(t)p(\sigma^{\omega}(t)) = p(t), \qquad t \in \mathbb{T}.$$

For example, $p(t) = \dfrac{c}{\sigma(t) - t}$ is a $1-$periodic function for any isolated time scale, for any constant c.

Hyers–Ulam Stability (HUS)

Stan Ulam, in **A Collection of Mathematical Problems**, 1960, posed the following question:

When is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation?

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Hyers–Ulam Stability (HUS): $q-$Difference Eqn

Equation

$$z^{\Delta}(t) - p(t)z(t) = 0, \qquad z^{\Delta}(t) := \frac{z(qt) - z(t)}{qt - t}, \tag{1}$$

has Hyers–Ulam stability if and only if there exists a constant $L > 0$ such that, for arbitrary $\varepsilon > 0$, if a function $\psi : \mathbb{T} \to \mathbb{C}$ satisfies

$$|\psi^{\Delta}(t) - p(t)\psi(t)| \leq \varepsilon, \qquad t \in \mathbb{T}, \tag{2}$$

then there exists a solution $z : \mathbb{T} \to \mathbb{C}$ of (1) such that $|\psi(t) - z(t)| \leq L\varepsilon$ for all $t \in \mathbb{T}$.

$q-$Exponential Function

Remark

For $q>1$, *let* $p(1)=p_1$ *for* $p_1 \in \mathbb{C}\setminus\left\{\frac{-1}{q-1}\right\}$. *Set*

$$e_p(t) := \prod_{j=0}^{\log_q t-1} \left(1+(q-1)q^j p(q^j)\right), \quad \textit{where} \quad \prod_{\ell=0}^{-1} f(\ell) \equiv 1.$$

For a $1-$*periodic function* p, *we have*

$$e_p(t) = [1+(q-1)p_1]^{\log_q t} \tag{3}$$

For notational convenience throughout the remainder of this section, define the base of the exponential function to be

$$A := 1+(q-1)p_1. \tag{4}$$

No Hyers–Ulam Stability (HUS)

Theorem (Anderson 2022)

Assume $q > 1$, $p(t) = \dfrac{p_1}{t}$ *is* $1-$*periodic, where* $p_1 \in \mathbb{C} \setminus \left\{ \frac{-1}{q-1} \right\}$, *and* $A = 1 + (q-1)p_1$. *Then,*

$$z^{\Delta}(t) - p(t)z(t) = 0$$

is not Hyers–Ulam stable.

Proof of No HUS

Let $\varepsilon > 0$ be a fixed arbitrary constant throughout the proof, which we split according to the following cases. Recall, $p(t) = \frac{p_1}{t}$, $A = 1 + (q-1)p_1$, and $e_p(t) = A^{\log_q t}$.

1. If $|A| = q$, then $z^{\Delta}(t) - p(t)z(t) = 0$ is not HUS.
2. If $|A| \neq q$, then $z^{\Delta}(t) - p(t)z(t) = 0$ is not HUS.

No HUS for $z^{\Delta}(t) - p(t)z(t) = 0$ (1)

Let

$$\psi(t) = (q-1)\varepsilon \log_q t A^{\log_q t - 1},$$

where $|A| = q$. Then ψ satisfies

$$|\psi^{\Delta}(t) - p(t)\psi(t)| = \varepsilon,$$

but

$$\begin{aligned} |\psi(t) - z(t)| &= \left| (q-1)\varepsilon \log_q t A^{\log_q t - 1} - cA^{\log_q t} \right| \\ &= t \left| (q-1)\varepsilon A^{-1} \log_q t - c \right| \longrightarrow \infty \end{aligned}$$

as $t \to \infty$ for any choice of $c \in \mathbb{C}$. Thus, in this case of $|A| = q > 1$, equation (1) is not Ulam stable.

No HUS for $z^{\Delta}(t) - p(t)z(t) = 0$ (1)

Let

$$\psi(t) = \frac{(q-1)\varepsilon\left(A^{\log_q t} - t\right)}{A-q},$$

where $|A| \neq q$. Then ψ again satisfies

$$|\psi^{\Delta}(t) - p(t)\psi(t)| = \varepsilon,$$

but

$$|\psi(t) - z(t)| = \left|\frac{(q-1)\varepsilon\left(A^{\log_q t} - t\right)}{A-q} - cA^{\log_q t}\right|, \quad t = q^n$$

$$= |A|^n \left|\left(\frac{(q-1)\varepsilon}{A-q} - c\right) - \frac{(q-1)\varepsilon q^n}{(A-q)A^n}\right| \longrightarrow \infty$$

as $t \to \infty$ for any choice of $c \in \mathbb{C}$. Thus, in this case of $|A| \neq q > 1$, equation (1) is also not HUS.

2−Periodic Function

The function $p : \mathbb{T} \to \mathbb{C}$ is 2−periodic if

$$p(t) = q^2 p\left(q^2 t\right), \qquad \forall t \in \mathbb{T}.$$

Let $p(1) = p_1$ and $p(q) = p_q$, for $p_1 \in \mathbb{C} \setminus \left\{\frac{-1}{q-1}\right\}$ and $p_q \in \mathbb{C} \setminus \left\{\frac{-1}{q(q-1)}\right\}$. Through iteration, one sees that p is given by

$$p(t) := \frac{1}{t} \begin{cases} p_1 & \text{if} \quad \log_q t \equiv 0 \pmod 2, \\ q p_q & \text{if} \quad \log_q t \equiv 1 \pmod 2. \end{cases}$$

2−Periodic Exponential Function

Let $p(1) = p_1$ and $p(q) = p_q$, for $p_1 \in \mathbb{C} \backslash \left\{ \frac{-1}{q-1} \right\}$ and $p_q \in \mathbb{C} \backslash \left\{ \frac{-1}{q(q-1)} \right\}$. By the 2-periodic nature of p given above, we have

$$e_p(t) = \begin{cases} A^{\frac{1}{2}\log_q t} B^{\frac{1}{2}\log_q t} & \text{if} \quad \log_q t \equiv 0 \pmod 2 \\ A^{\frac{1}{2}(\log_q t+1)} B^{\frac{1}{2}(\log_q t-1)} & \text{if} \quad \log_q t \equiv 1 \pmod 2, \end{cases}$$

where $A = [1 + (q-1)p_1]$ and $B = [1 + (q-1)qp_q]$.

No HUS for 2−Periodic Eqn

Theorem (Anderson 2022)

Assume $q > 1$, $p_1 \in \mathbb{C} \setminus \left\{ \frac{-1}{q-1} \right\}$, $p_q \in \mathbb{C} \setminus \left\{ \frac{-1}{q(q-1)} \right\}$. *For*

$$p(t) := \frac{1}{t} \begin{cases} p_1 & \text{if} \quad \log_q t \equiv 0 \pmod 2 \\ q p_q & \text{if} \quad \log_q t \equiv 1 \pmod 2, \end{cases}$$

$z^{\Delta}(t) - p(t)z(t) = 0$ *is not HUS.*

The proof is split according to the following cases.

1. If $|AB| = q^2$, then (1) is not HUS.
2. If $|AB| \neq q^2$, then (1) is not HUS.

HUS for 1−Periodic Eqn, if ...

Theorem (2022)

Let $\mathbb{T} = \{t_n\}_{n=0}^{\infty}$ *be an isolated time scale with graininess function* $\mu(t) = t_{n+1} - t_n > 0$ *such that* $\lim_{n\to\infty} t_n = \infty$. *Let* $p(t) = \frac{\mu_0 p_0}{\mu(t)}$ *for all* $t = t_n \in \mathbb{T}$*; videlicet,* p *is* $1-$*periodic, where* $p_0 \in \mathbb{C} \setminus \left\{ -\frac{1}{\mu_0} \right\}$. *If there exists a positive constant* $\mu_{\max} \in (0, \infty)$ *such that*

$$\mu(t_n) \leq \mu_{\max},$$

then (1) is HUS if and only if $\rho = |1 + \mu_0 p_0| \neq 1$*, with Ulam constant* $L = \frac{\mu_{\max}}{|1-\rho|}$.

No HUS for 1−Periodic Eqn, if ...

Theorem (2022)

Let $\mathbb{T} = \{t_n\}_{n=0}^{\infty}$ *be an isolated time scale. Let* $p(t) = \frac{\mu_0 p_0}{\mu(t)}$ *for all* $t = t_n \in \mathbb{T}$*; that is to say,* p *is* $1-$*periodic. If there exist real constants* $a > 0$ *and* $b \geq 0$ *such that*

$$\mu(t_n) \geq an + b,$$

then (1) is not HUS.

Remark

The linear growth condition above can be significantly weakened to the condition

$$\mu(t_n) \geq an^{\gamma} + b,$$

for any real $\gamma > 0$*.*

2−Periodic Function, Exponential Function

A 2−periodic function p is given by

$$p(t_n) := \frac{1}{\mu(t_n)} \begin{cases} \mu_0 p_0 & \text{if} \quad n \equiv 0 \pmod 2 \\ \mu_1 p_1 & \text{if} \quad n \equiv 1 \pmod 2, \end{cases}$$

and the corresponding exponential function is

$$e_p(t) = \begin{cases} A^{\frac{n}{2}} B^{\frac{n}{2}} & \text{if} \quad n \equiv 0 \pmod 2 \\ A^{\frac{n+1}{2}} B^{\frac{n-1}{2}} & \text{if} \quad n \equiv 1 \pmod 2, \end{cases}$$

where $A = [1 + \mu_0 p_0]$ and $B = [1 + \mu_1 p_1]$.

2−Periodic HUS for isolated time scale $\mathbb{T} = \{t_n\}_{n=0}^{\infty}$

Theorem (2022)

Let $p(t_k) = p_k$, $\mu(t_k) = \mu_k$, *and* $p_k \in \mathbb{C}\setminus\left\{\frac{-1}{\mu_k}\right\}$ *for* $k = 0, 1$. *If* μ *satisfies*

$$\mu(t_n) \leq \mu_{\max}$$

and p is a 2−*periodic function, then (1) is HUS stable if and only if*

$$|AB| = |(1 + \mu_0 p_0)(1 + \mu_1 p_1)| \neq 1$$

with Ulam constant

$$L = \max\{|A| + 1, |B| + 1\}\frac{\mu_{\max}}{||AB| - 1|}.$$

If $\mu(t_n) \geq an + b$ *holds for some* a, b*, then (1) is not HUS.*

Example: $h-$difference equations

Let $h > 0$, and set $\mathbb{T} = \{0, h, 2h, 3h, \ldots\}$. If p is a $1-$periodic function on $\mathbb{T}$, then $p(t+h) = p(t)$ for all $t \in \mathbb{T}$, so that $p(t) \equiv p$ is constant, due to the constant graininess $\mu(t) \equiv h$. Then (1) is HUS for $p \in \mathbb{C} \setminus \left\{-\frac{1}{h}\right\}$ if and only if $|1 + hp| \neq 1$, with best Ulam constant

$$L = \frac{h}{|1 - |1 + hp||} = \frac{1}{|\mathrm{Re}_h(p)|}.$$

Clearly, $h = \mu_{\max}$ is bounded.

Example: 2 step sizes

Let η, $\tau > 0$ be two step sizes, and let the isolated time scale $\mathbb{T}$ constructed with them be denoted by

$$\mathbb{T}_{\eta,\tau} := \{0, \eta, \eta+\tau, (\eta+\tau)+\eta, 2(\eta+\tau), 2(\eta+\tau)+\eta, \cdots\}.$$

Then the derivative is

$$z^{\Delta}(t) := \begin{cases} \frac{z(t+\eta)-z(t)}{\eta} & : \frac{t}{\eta+\tau} \in \mathbb{Z} \\ \frac{z(t+\tau)-z(t)}{\tau} & : \frac{t-\eta}{\eta+\tau} \in \mathbb{Z} \end{cases}$$

for $t \in \mathbb{T}_{\eta,\tau}$. For the periodic coefficient case, $\mu(t_n) \leq \mu_{\max} := \max\{\eta, \tau\}$. If p is $1-$periodic, then

$$p(t_n) = p_0 \begin{cases} 1 & \text{if} \quad n \equiv 0 \pmod 2 \\ \frac{\eta}{\tau} & \text{if} \quad n \equiv 1 \pmod 2 \end{cases}$$

for $p_0 \in \mathbb{C} \backslash \{-\frac{1}{\eta}\}$. By the theorem, (1) is Ulam stable if and only if $\rho = |1+\eta p_0| \neq 1$, with Ulam constant given by $L = \frac{\max\{\eta,\tau\}}{|1-\rho|}$.

2 step sizes, 2−periodic case

If p is 2−periodic, then

$$p(t_n) = \begin{cases} p_0 & \text{if} \quad n \equiv 0 \pmod 2 \\ p_1 & \text{if} \quad n \equiv 1 \pmod 2 \end{cases}$$

for $p_0 \in \mathbb{C}\backslash\{-\frac{1}{\eta}\}$ and $p_1 \in \mathbb{C}\backslash\{-\frac{1}{\tau}\}$. By the theorem, (1) is Ulam stable if and only if $|(1+\eta p_0)(1+\tau p_1)| \neq 1$, with Ulam constant given by

$$L = \max\{|1+\eta p_0|+1, |1+\tau p_1|+1\} \frac{\max\{\eta, \tau\}}{||(1+\eta p_0)(1+\tau p_1)|-1|}.$$

Example: Triangular equation

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$, and let

$$\mathbb{T} = \left\{ \frac{n(n+1)}{2} \right\}_{n=0}^{\infty} = \{0, 1, 3, 6, 10, \ldots\},$$

the set of triangular numbers. It follows that

$$t_{n+1} - t_n = \mu(t_n) = n + 1, \quad n \in \mathbb{N}_0.$$

If p is $1-$periodic on $\mathbb{T}$, then $\left(\frac{n+2}{n+1}\right) p(t_{n+1}) = p(t_n)$. Thus,

$$p(t) = \frac{p_0}{\mu(t)} \implies p(t_n) = \frac{p_0}{n+1}, \quad t \in \mathbb{T}.$$

The exponential function is $e_p(t_n) = (1 + p_0)^n$ for $p_0 \in \mathbb{C} \setminus \{-1\}$. Then, (1) is not HUS, since $\mu(t_n) = an + b$ for $a = 1 = b$ for all $n \in \mathbb{N}_0$.

Example: Fibonacci equation

Consider the set of Fibonacci numbers

$$\mathbb{T} = \left\{ \frac{\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}}{2^{n+1}\sqrt{5}} \right\}_{n=1}^{\infty} = \{1, 2, 3, 5, 8, 13, \ldots\},$$

where we have omitted the first 1 to avoid the redundancy of two consecutive 1s. It follows that

$$t_{n+1} - t_n = \mu(t_n) = t_{n-1}, \quad n \in \{2, 3, 4, \ldots\}, \quad \mu(t_1) = 1.$$

If p is 1−periodic on $\mathbb{T}$, then $\left(\frac{t_n}{t_{n-1}}\right) p(t_{n+1}) = p(t_n)$. Thus,

$$p(t) = \frac{p_1}{\mu(t)} \implies p(t_n) = \frac{p_1}{t_{n-1}}, \quad t \in \mathbb{T}, \quad t_0 = 1.$$

The exponential function is $e_p(t_n) = (1 + p_1)^{n-1}$ for $p_1 \in \mathbb{C} \setminus \{-1\}$. Then, (1) is not HUS, since $\mu(t_n) = t_{n-1} \geq \frac{1}{2}n$ for all $n \in \mathbb{N}_0$.

Example: Harmonic equation

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$, and let

$$\mathbb{T} = \left\{ H_n = \sum_{j=1}^{n} \frac{1}{j} \right\}_{n=0}^{\infty} = \left\{ 0, 1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12} \ldots \right\}, \quad n \in \mathbb{N}_0,$$

the set of harmonic numbers. It follows that

$$H_{n+1} - H_n = \mu(H_n) = \frac{1}{n+1} \leq 1, \quad n \in \mathbb{N}_0.$$

If p is $1-$periodic, then $\left(\frac{n+1}{n+2}\right) p(H_{n+1}) = p(H_n)$. Thus,

$$p(t) = \frac{p_0}{\mu(t)} \implies p(H_n) = (n+1)p_0, \quad t \in \mathbb{T}.$$

The exponential is $e_p(H_n) = (1 + p_0)^n$ for $p_0 \in \mathbb{C} \setminus \{-1\}$. Then, (1) is HUS if and only if $|1 + p_0| \neq 1$.

Example: Harmonic equation

Additional work shows that (1) is HUS with Ulam constant $L = \ln\left(\frac{\rho}{\rho-1}\right)$ for $\rho > 1$, $1 + p_0 = \rho e^{i\theta}$, and $e_p(H_n) = \rho^n e^{in\theta}$, and (1) is HUS with Ulam constant

$$L = \begin{cases} \max\left\{\ln\left(\frac{1-\rho}{\rho}\right), \rho + \frac{1}{2}\right\} & \text{if} \quad 0 < \rho \le \frac{1}{2} \\ \max\left\{\ln\left(\frac{\rho}{1-\rho}\right), \rho + \frac{1}{2}, \frac{1}{3} + \frac{\rho}{2} + \rho^2\right\} & \text{if} \quad \frac{1}{2} \le \rho < 1 \end{cases}$$

for $0 < \rho < 1$, $1 + p_0 = \rho e^{i\theta}$, and $e_p(H_n) = \rho^n e^{in\theta}$.

Thanks for Listening!