WHEN AM I GOING TO USE THIS?

Right now, in a classroom somewhere in the world, a student is mouthing off to her math teacher. The teacher has just asked her to spend a substantial portion of her weekend computing a list of thirty definite integrals.

There are other things the student would rather do. There is, in fact, hardly anything she would *not* rather do. She knows this quite clearly, because she spent a substantial portion of the previous weekend computing a different—but not *very* different—list of thirty definite integrals. She doesn't see the point, and she tells her teacher so. And at some point in this conversation, the student is going to ask the question the teacher fears most:

"When am I going to use this?"

Now the math teacher is probably going to say something like:

"I know this seems dull to you, but remember, you don't know what career you'll choose—you may not see the relevance now, but you might go into a field where it'll be really important that you know how to compute definite integrals quickly and correctly by hand."

This answer is seldom satisfying to the student. That's because it's a lie. And the teacher and the student both know it's a lie. The number of adults who will ever make use of the integral of $(1 - 3x + 4x^2)^{-2}\,dx$, or the formula for the cosine of 3θ, or synthetic division of polynomials, can be counted on a few thousand hands.

The lie is not very satisfying to the teacher, either. I should know: in my many years as a math professor I've asked many hundreds of college students to compute lists of definite integrals.

Fortunately, there's a better answer. It goes something like this:

"Mathematics is not just a sequence of computations to be carried out by rote until your patience or stamina runs out—although it might seem that way from what you've been taught in courses called *mathematics*. Those integrals are to mathematics as weight training and calisthenics are to soccer. If you want to play soccer—I mean, *really play*, at a competitive level—you've got to do a lot of boring, repetitive, apparently pointless drills. Do professional players ever *use* those drills? Well, you won't see anybody on the field curling a weight or zigzagging between traffic cones. But you do see players using the strength, speed, insight, and flexibility they built up by doing those drills, week after tedious week. Learning those drills is part of learning soccer.

"If you want to play soccer for a living, or even make the varsity team, you're going to be spending lots of boring weekends on the practice field. There's no other way. But now here's the good news. If the drills are too much for you to take, you can still play for fun, with friends. You can enjoy the thrill of making a slick pass between defenders or scoring from distance just as much as a pro athlete does. You'll be healthier and happier than you would be if you sat home watching the professionals on TV.

"Mathematics is pretty much the same. You may not be aiming for a mathematically oriented career. That's fine—most people aren't. But you can still do math. You probably already *are* doing math, even if you don't call it that. Math is woven into the way we reason. And math makes you better at things. Knowing mathematics is like wearing a pair of X-ray specs that reveal hidden structures underneath the messy and chaotic surface of the world. Math is a science of not being wrong about things, its techniques and habits hammered out by centuries of hard work and argument. With the tools of mathematics in hand, you can understand the world in a deeper, sounder, and more meaningful way. All you need is a coach, or even just a book, to teach you the rules and some basic tactics. I will be your coach. I will show you how."

For reasons of time, this is seldom what I actually say in the classroom. But in a book, there's room to stretch out a little more. I hope to back up the grand claims I just made by showing you that the problems we think about every day—problems of politics, of medicine, of commerce, of theology—are shot through with mathematics. Understanding this gives you access to insights accessible by no other means.

Even if I did give my student the full inspirational speech, she might—if she is really sharp—remain unconvinced.

"That sounds good, Professor," she'll say. "But it's pretty abstract. You say that with mathematics at your disposal you can get things right you'd otherwise get wrong. But what kind of things? Give me an *actual example*."

And at that point I would tell her the story of Abraham Wald and the missing bullet holes.

ABRAHAM WALD AND THE MISSING BULLET HOLES

This story, like many World War II stories, starts with the Nazis hounding a Jew out of Europe and ends with the Nazis regretting it. Abraham Wald was born in 1902 in what was then the city of Klausenburg in what was then the Austro-Hungarian Empire. By the time Wald was a teenager, one World War was in the books and his hometown had become Cluj, Romania. He was the grandson of a rabbi and the son of a kosher baker, but the younger Wald was a mathematician almost from the start. His talent for the subject was quickly recognized, and he was admitted to study mathematics at the University of Vienna, where he was drawn to subjects abstract and recondite even by the standards of pure mathematics: set theory and metric spaces.

But when Wald's studies were completed, it was the mid-1930s, Austria was deep in economic distress, and there was no possibility that a foreigner could be hired as a professor in Vienna. Wald was rescued by a job offer from Oskar Morgenstern. Morgenstern would later immigrate to the United States and help invent game theory, but in 1933 he was the director of the Austrian Institute for Economic Research, and he hired Wald at a small salary to do mathematical odd jobs. That turned out to be a good move for Wald: his experience in economics got him a fellowship offer at the Cowles Commission, an economic institute then located in Colorado Springs. Despite the ever-worsening political situation, Wald was reluctant to take a step that would lead him away from pure mathematics for good. But then the Nazis conquered Austria, making Wald's decision substantially easier. After just a few months in Colorado, he was offered a professorship of statistics at Columbia; he packed up once again and moved to New York.

And that was where he fought the war.

The Statistical Research Group (SRG), where Wald spent much of World War II, was a classified program that yoked the assembled might of American statisticians to the war effort—something like the Manhattan Project, except the weapons being developed were equations, not explosives. And the SRG was actually *in* Manhattan, at 401 West 118th Street in Morningside Heights, just a block away from Columbia University. The building now houses Columbia faculty apartments and some doctor's offices, but in 1943 it was the buzzing, sparking nerve center of wartime math. At the Applied Mathematics Group–Columbia, dozens of young women bent over Marchant desktop calculators were calculating formulas for the optimal curve a fighter should trace out through the air in order to keep an enemy plane in its gunsights. In another apartment, a team of researchers from Princeton was developing protocols for strategic bombing. And Columbia's wing of the atom bomb project was right next door.

But the SRG was the most high-powered, and ultimately the most influential, of any of these groups. The atmosphere combined the intellectual openness and intensity of an academic department with the shared sense of purpose that comes only with high stakes. "When we made recommendations," W. Allen Wallis, the director, wrote, "frequently things happened. Fighter planes entered combat with their machine guns loaded according to Jack Wolfowitz's* recommendations about mixing types of ammunition, and maybe the pilots came back or maybe they didn't. Navy planes launched rockets whose propellants had been accepted by Abe Girshick's sampling-inspection plans, and maybe the rockets exploded and destroyed our own planes and pilots or maybe they destroyed the target."

* Paul's dad.

The mathematical talent at hand was equal to the gravity of the task. In Wallis's words, the SRG was "the most extraordinary group of statisticians ever organized, taking into account both number and quality." Frederick Mosteller, who would later found Harvard's statistics department, was there. So was Leonard Jimmie Savage, the pioneer of decision theory and great advocate of the field that came to be called Bayesian statistics.* Norbert Wiener, the MIT mathematician and the creator of cybernetics, dropped by from time to time. This was a group where Milton Friedman, the future Nobelist in economics, was often the fourth-smartest person in the room.

The *smartest* person in the room was usually Abraham Wald. Wald had been Allen Wallis's teacher at Columbia, and functioned as a kind of mathematical eminence to the group. Still an "enemy alien," he was not technically allowed to see the classified reports he was producing; the joke around SRG was that the secretaries were required to pull each sheet of notepaper out of his hands as soon as he was finished writing on it. Wald was, in some ways, an unlikely participant. His inclination, as it always had been, was toward abstraction, and away from direct applications. But his motivation to use his talents against the Axis was obvious. And when you needed to turn a vague idea into solid mathematics, Wald was the person you wanted at your side.

So here's the question. You don't want your planes to get shot down by enemy fighters, so you armor them. But armor makes the plane heavier, and heavier planes are less maneuverable and use more fuel. Armoring the planes too much is a problem; armoring the planes too little is a problem. Somewhere in between there's an optimum. The reason you

* Savage was almost totally blind, able to see only out of one corner of one eye, and at one point spent six months living only on pemmican in order to prove a point about Arctic exploration. Just thought that was worth mentioning.

have a team of mathematicians socked away in an apartment in New York City is to figure out where that optimum is.

The military came to the SRG with some data they thought might be useful. When American planes came back from engagements over Europe, they were covered in bullet holes. But the damage wasn't uniformly distributed across the aircraft. There were more bullet holes in the fuselage, not so many in the engines.

Section of plane	Bullet holes per square foot
Engine	1.11
Fuselage	1.73
Fuel system	1.55
Rest of the plane	1.8

The officers saw an opportunity for efficiency; you can get the same protection with less armor if you concentrate the armor on the places with the greatest need, where the planes are getting hit the most. But exactly how much more armor belonged on those parts of the plane? That was the answer they came to Wald for. It wasn't the answer they got.

The armor, said Wald, doesn't go where the bullet holes are. It goes where the bullet holes *aren't:* on the engines.

Wald's insight was simply to ask: where are the missing holes? The ones that would have been all over the engine casing, if the damage had been spread equally all over the plane? Wald was pretty sure he knew. The missing bullet holes were on the missing planes. The reason planes were coming back with fewer hits to the engine is that planes that got hit in the engine weren't coming back. Whereas the large number of planes returning to base with a thoroughly Swiss-cheesed fuselage is pretty strong evidence that hits to the fuselage can (and therefore should) be tolerated. If you go to the recovery room at the hospital, you'll see a lot more people with bullet holes in their legs than people with bullet holes in their chests. But that's not because people don't get shot in the chest; it's because the people who get shot in the chest don't recover.

Here's an old mathematician's trick that makes the picture perfectly clear: *set some variables to zero.* In this case, the variable to tweak is the probability that a plane that takes a hit to the engine manages to stay in the air. Setting that probability to zero means a single shot to the engine is guaranteed to bring the plane down. What would the data look like then? You'd have planes coming back with bullet holes all over the wings, the fuselage, the nose—but none at all on the engine. The military analyst has two options for explaining this: either the German bullets just happen to hit every part of the plane but one, or the engine is a point of total vulnerability. Both stories explain the data, but the latter makes a lot more sense. The armor goes where the bullet holes aren't.

Wald's recommendations were quickly put into effect, and were still being used by the navy and the air force through the wars in Korea and Vietnam. I can't tell you exactly how many American planes they saved, though the data-slinging descendants of the SRG inside today's military no doubt have a pretty good idea. One thing the American defense establishment has traditionally understood very well is that countries don't win wars just by being braver than the other side, or freer, or slightly preferred by God. The winners are usually the guys who get 5% fewer of their planes shot down, or use 5% less fuel, or get 5% more nutrition into their infantry at 95% of the cost. That's not the stuff war movies are made of, but it's the stuff wars are made of. And there's math every step of the way.

Why did Wald see what the officers, who had vastly more knowledge and understanding of aerial combat, couldn't? It comes back to his math-trained habits of thought. A mathematician is always asking, "What assumptions are you making? And are they justified?" This can be annoying. But it can also be very productive. In this case, the officers were making an assumption unwittingly: that the planes that came back were a random sample of all the planes. If that were true, you could draw conclusions about the distribution of bullet holes on all the planes by examining the distribution of bullet holes on only the surviving planes. Once you recognize that you've been making that hypothesis, it only takes a moment to realize it's dead wrong; there's no reason at all to expect the planes to have an equal likelihood of survival no matter where they get hit. In a piece of mathematical lingo we'll come back to in chapter 15, the rate of survival and the location of the bullet holes are *correlated*.

Wald's other advantage was his tendency toward abstraction. Wolfowitz, who had studied under Wald at Columbia, wrote that the problems he favored were "all of the most abstract sort," and that he was "always ready to talk about mathematics, but uninterested in popularization and special applications."

Wald's personality made it hard for him to focus his attention on applied problems, it's true. The details of planes and guns were, to his eye, so much upholstery—he peered right through to the mathematical struts and nails holding the story together. Sometimes that approach can lead you to ignore features of the problem that really matter. But it also lets you see the common skeleton shared by problems that look very different on the surface. Thus you have meaningful experience even in areas where you appear to have none.

To a mathematician, the structure underlying the bullet hole problem is a phenomenon called *survivorship bias*. It arises again and again, in all kinds of contexts. And once you're familiar with it, as Wald was, you're primed to notice it wherever it's hiding.

Like mutual funds. Judging the performance of funds is an area where you don't want to be wrong, even by a little bit. A shift of 1% in annual growth might be the difference between a valuable financial asset and a dog. The funds in Morningstar's Large Blend category, whose mutual funds invest in big companies that roughly represent the S&P 500, look like the former kind. The funds in this class grew an average of 178.4% between 1995 and 2004: a healthy 10.8% per year.* Sounds like you'd do well, if you had cash on hand, to invest in those funds, no?

Well, no. A 2006 study by Savant Capital shone a somewhat colder light on those numbers. Think again about how Morningstar generates its number. It's 2004, you take all the funds classified as Large Blend, and you see how much they grew over the last ten years.

But something's missing: *the funds that aren't there*. Mutual funds don't live forever. Some flourish, some die. The ones that die are, by and large, the ones that don't make money. So judging a decade's worth of mutual funds by the ones that still exist at the end of the ten years is like judging our pilots' evasive maneuvers by counting the bullet holes in the planes that come back. What would it mean if we never found more than one bullet hole per plane? Not that our pilots are brilliant at dodging enemy fire, but that the planes that got hit twice went down in flames.

The Savant study found that if you included the performance of the dead funds together with the surviving ones, the rate of return dropped down to 134.5%, a much more ordinary 8.9% per year. More recent research backed that up: a comprehensive 2011 study in the *Review of Finance* covering nearly 5,000 funds found that the excess return rate of the 2,641 survivors is about 20% higher than the same figure recomputed to include the funds that didn't make it. The size of the survivorship effect might have surprised investors, but it probably wouldn't have surprised Abraham Wald.

MATHEMATICS IS THE EXTENSION OF COMMON SENSE BY OTHER MEANS

At this point my teenaged interlocutor is going to stop me and ask, quite reasonably: Where's the math? Wald was a mathematician, that's true, and it can't be denied that his solution to the problem of the bullet holes was ingenious, but what's mathematical about it? There was no trig identity to be seen, no integral or inequality or formula.

First of all: Wald did use formulas. I told the story without them, because this is just the introduction. When you write a book explaining human reproduction to preteens, the introduction stops short of the really hydraulic stuff about how babies get inside Mommy's tummy. Instead, you start with something more like "Everything in nature changes; trees lose their leaves in winter only to bloom again in spring; the humble caterpillar enters its chrysalis and emerges as a magnificent butterfly. You are part of nature too, and . . ."

That's the part of the book we're in now.

But we're all adults here. Turning off the soft focus for a second, here's what a sample page of Wald's actual report looks like:

* To be fair, the S&P 500 index itself did even better, gaining 212.5% over the same period.

lower bound to the Q_i could be obtained. The assumption here is that the decrease from q_i to q_{i+1} lies between definite limits. Therefore, both an upper and lower bound for the Q_i can be obtained.

We assume that

$$\lambda_1 q_i \leq q_{i+1} \leq \lambda_2 q_i ,$$

where $\lambda_1 < \lambda_2 < 1$ and such that the expression

$$\sum_{j=1}^{n} \frac{a_j}{\lambda_1^{\frac{j(j-1)}{2}}} < 1 - a_o \qquad \text{(A)}$$

is satisfied.

The exact solution is tedious but close approximations to the upper and lower bounds to the Q_i for $i < n$ can be obtained by the following procedure. The set of hypothetical data used is

$a_o = .780$	$a_3 = .010$
$a_1 = .070$	$a_4 = .005$
$a_2 = .040$	$a_5 = .005$
$\lambda_1 = .80$	$\lambda_2 = .90$

Condition A is satisfied, since by substitution

$$.07 + \frac{.04}{.8} + \frac{.01}{(.8)^3} + \frac{.005}{(.8)^6} + \frac{.005}{(.8)^{10}} = .20529 ,$$

which is less than

$$1 - a_o = .22 .$$

THE LOWER LIMIT OF Q_i

The first step is to solve equation 66. This involves the solution of the following four equations for positive roots g_o, g_1, g_2, g_3.

I hope that wasn't too shocking.

Still, the real *idea* behind Wald's insight doesn't require any of the formalism above. We've already explained it, using no mathematical notation of any kind. So my student's question stands. What makes that math? Isn't it just common sense?

Yes. Mathematics *is* common sense. On some basic level, this is clear. How can you explain to someone why adding seven things to five things yields the same result as adding five things to seven? You can't: that fact is baked into our way of thinking about combining things together. Mathematicians like to give names to the phenomena our common sense describes: instead of saying, "*This* thing added to *that* thing is the *same* thing as *that* thing added to *this* thing," we say, "Addition is commutative." Or, because we like our symbols, we write:

For any choice of a and b, a + b = b + a.

Despite the official-looking formula, we are talking about a fact instinctively understood by every child.

Multiplication is a slightly different story. The formula looks pretty similar:

For any choice of a and b, a × b = b × a.

The mind, presented with this statement, does not say "no duh" quite as instantly as it does for addition. Is it "common sense" that two sets of six things amount to the same as six sets of two?

Maybe not; but it can *become* common sense. Here's my earliest mathematical memory. I'm lying on the floor in my parents' house, my cheek pressed against the shag rug, looking at the stereo. Very probably I am listening to side two of the Beatles' Blue Album. Maybe I'm six. This is the seventies, and therefore the stereo is encased in a pressed wood panel, which has a rectangular array of airholes punched into the side. Eight holes across, six holes up and down. So I'm lying there, looking at the airholes. The six rows of holes. The eight columns of holes. By focusing my gaze in and out I could make my mind flip back and forth between seeing the rows and seeing the columns. Six rows with eight holes each. Eight columns with six holes each.

And then I had it—eight groups of six were the same as six groups of eight. Not because it was a rule I'd been told, but because it could not be any other way. The number of holes in the panel was the number of holes in the panel, no matter which way you counted them.