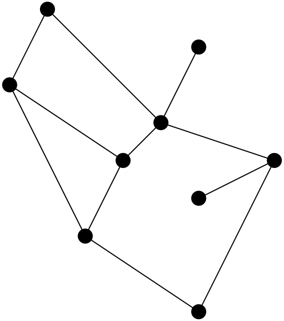
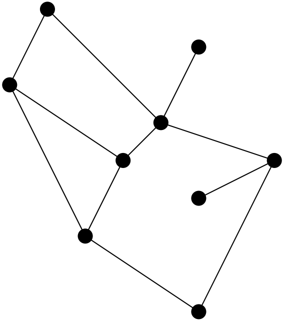
Theorem

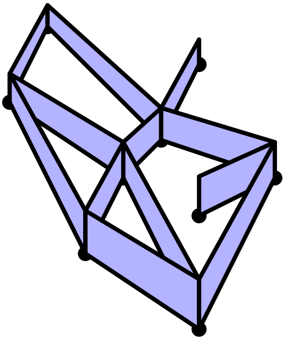
Suppose we have a simple connected planar graph.
Let V be the number of vertices and E the number of edges.
Then $3V - 6 \geq E$.



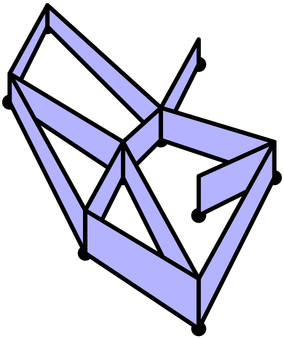
- Let's pretend our graph is the floor plan of a house.
- So each "room," counting the outside, is a face.
- Suppose you have to paint the walls (inside and out).
- How many walls will you have to paint?



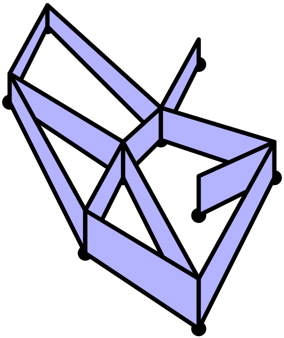
- Let's pretend our graph is the floor plan of a house.
- So each "room," counting the outside, is a face.
- Suppose you have to paint the walls (inside and out).
- How many walls will you have to paint?



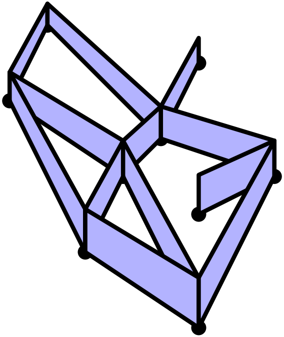
- Let’s pretend our graph is the floor plan of a house.
- So each “room,” counting the outside, is a face.
- Suppose you have to paint the walls (inside and out).
- How many walls will you have to paint?



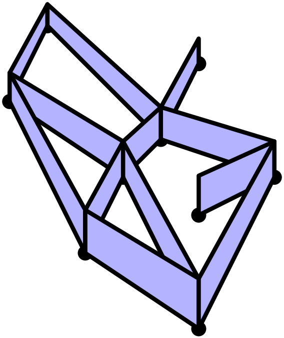
- Let's pretend our graph is the floor plan of a house.
- So each "room," counting the outside, is a face.
- Suppose you have to paint the walls (inside and out).
- How many walls will you have to paint?



- Let's pretend our graph is the floor plan of a house.
- So each "room," counting the outside, is a face.
- Suppose you have to paint the walls (inside and out).
- How many walls will you have to paint?



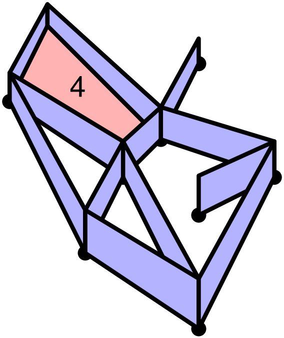
- Let’s pretend our graph is the floor plan of a house.
- So each “room,” counting the outside, is a face.
- Suppose you have to paint the walls (inside and out).
- How many walls will you have to paint?



Answer #1:

Every room has at least three walls, so the number of walls to paint is at least $3F$.

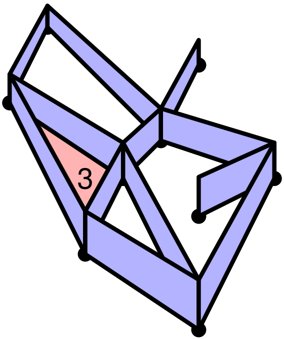
$$W \geq 3F.$$



Answer #1:

Every room has at least three walls, so the number of walls to paint is at least $3F$.

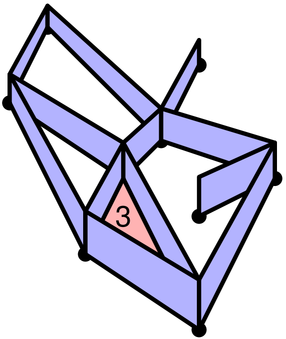
$$W \geq 3F.$$



Answer #1:

Every room has at least three walls, so the number of walls to paint is at least $3F$.

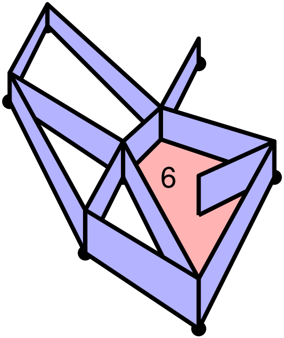
$$W \geq 3F.$$



Answer #1:

Every room has at least three walls, so the number of walls to paint is at least $3F$.

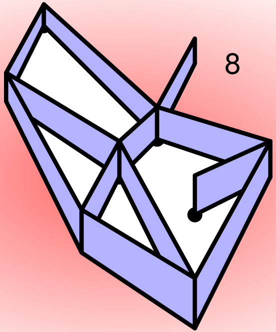
$$W \geq 3F.$$



Answer #1:

Every room has at least three walls, so the number of walls to paint is at least $3F$.

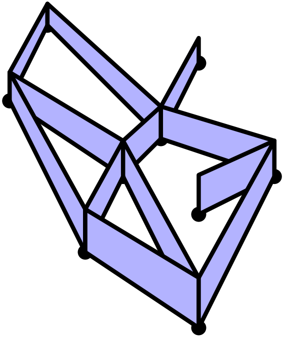
$$W \geq 3F.$$



Answer #1:

Every room has at least three walls, so the number of walls to paint is at least $3F$.

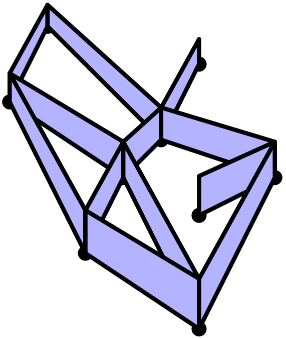
$$W \geq 3F.$$



Answer #1:

Every room has at least three walls, so the number of walls to paint is at least $3F$.

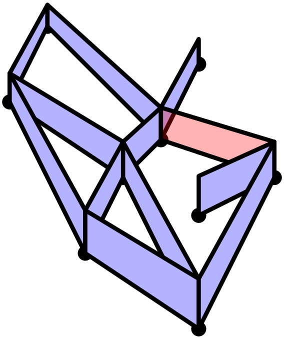
$$W \geq 3F.$$



Answer #2:

Every edge has two sides, so the number of walls to paint equals $2E$.

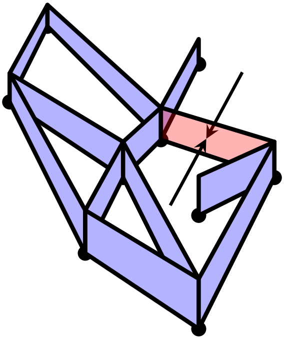
$$W = 2E.$$



Answer #2:

Every edge has two sides, so the number of walls to paint equals $2E$.

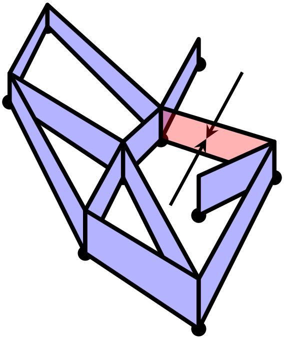
$$W = 2E.$$



Answer #2:

Every edge has two sides, so the number of walls to paint equals $2E$.

$$W = 2E.$$



Answer #2:

Every edge has two sides, so the number of walls to paint equals $2E$.

$$W = 2E.$$

Thus $2E = W$ and $W \geq 3F$, so

$2E \geq 3F$.

By Euler's Formula, $V - E + F = 2$ so $F = 2 - V + E$.

$$\begin{aligned}
\text{Thus} \quad && 2E &\geq 3F \\
\text{so} \quad && 2E &\geq 3(2 - V + E) \\
\text{so} \quad && 2E &\geq 6 - 3V + 3E \\
\text{Subtract } 2E \text{ from both sides:} \quad && 0 &\geq 6 - 3V + E \\
\text{Add } 3V \text{ to both sides:} \quad && 3V &\geq 6 + E \\
\text{Subtract 6 from both sides:} \quad && 3V - 6 &\geq E
\end{aligned}$$

Theorem

If a simple connected graph is planar, then $3V - 6 \geq E$.

Thus $2E = W$ and $W \geq 3F$, so

$2E \geq 3F$.

By Euler's Formula, $V - E + F = 2$ so $F = 2 - V + E$.

$$\begin{aligned} \text{Thus} \quad & 2E \geq 3F \\ \text{so} \quad & 2E \geq 3(2 - V + E) \\ \text{so} \quad & 2E \geq 6 - 3V + 3E \\ \text{Subtract } 2E \text{ from both sides:} \quad & 0 \geq 6 - 3V + E \\ \text{Add } 3V \text{ to both sides:} \quad & 3V \geq 6 + E \\ \text{Subtract 6 from both sides:} \quad & 3V - 6 \geq E \end{aligned}$$

Theorem

If a simple connected graph is planar, then $3V - 6 \geq E$.

Thus $2E = W$ and $W \geq 3F$, so

$2E \geq 3F$.

By Euler's Formula, $V - E + F = 2$ so $F = 2 - V + E$.

$$\begin{aligned} \text{Thus} \quad & 2E \geq 3F \\ \text{so} \quad & 2E \geq 3(2 - V + E) \\ \text{so} \quad & 2E \geq 6 - 3V + 3E \\ \text{Subtract } 2E \text{ from both sides:} \quad & 0 \geq 6 - 3V + E \\ \text{Add } 3V \text{ to both sides:} \quad & 3V \geq 6 + E \\ \text{Subtract 6 from both sides:} \quad & 3V - 6 \geq E \end{aligned}$$

Theorem

If a simple connected graph is planar, then $3V - 6 \geq E$.

Thus $2E = W$ and $W \geq 3F$, so

$2E \geq 3F$.

By Euler's Formula, $V - E + F = 2$ so $F = 2 - V + E$.

$$
\begin{aligned}
\text{Thus} \quad & 2E \geq 3F \\
\text{so} \quad & 2E \geq 3(2 - V + E) \\
\text{so} \quad & 2E \geq 6 - 3V + 3E \\
\text{Subtract } 2E \text{ from both sides:} \quad & 0 \geq 6 - 3V + E \\
\text{Add } 3V \text{ to both sides:} \quad & 3V \geq 6 + E \\
\text{Subtract 6 from both sides:} \quad & 3V - 6 \geq E
\end{aligned}
$$

Theorem

If a simple connected graph is planar, then $3V - 6 \geq E$.

Thus $2E = W$ and $W \geq 3F$, so

$2E \geq 3F$.

By Euler's Formula, $V - E + F = 2$ so $F = 2 - V + E$.

Thus	$2E \geq 3F$
so	$2E \geq 3(2 - V + E)$
so	$2E \geq 6 - 3V + 3E$
Subtract $2E$ from both sides:	$0 \geq 6 - 3V + E$
Add $3V$ to both sides:	$3V \geq 6 + E$
Subtract 6 from both sides:	$3V - 6 \geq E$

Theorem

If a simple connected graph is planar, then $3V - 6 \geq E$.

Thus $2E = W$ and $W \geq 3F$, so

$2E \geq 3F$.

By Euler's Formula, $V - E + F = 2$ so $F = 2 - V + E$.

$$\begin{aligned}
\text{Thus} \qquad 2E &\geq 3F \\
\text{so} \qquad 2E &\geq 3(2 - V + E) \\
\text{so} \qquad 2E &\geq 6 - 3V + 3E \\
\text{Subtract } 2E \text{ from both sides:} \qquad 0 &\geq 6 - 3V + E \\
\text{Add } 3V \text{ to both sides:} \qquad 3V &\geq 6 + E \\
\text{Subtract 6 from both sides:} \qquad 3V - 6 &\geq E
\end{aligned}$$

Theorem

If a simple connected graph is planar, then $3V - 6 \geq E$.

Thus $2E = W$ and $W \geq 3F$, so

$2E \geq 3F$.

By Euler's Formula, $V - E + F = 2$ so $F = 2 - V + E$.

$$\begin{aligned}
\text{Thus} \quad && 2E &\geq 3F \\
\text{so} \quad && 2E &\geq 3(2 - V + E) \\
\text{so} \quad && 2E &\geq 6 - 3V + 3E \\
\text{Subtract } 2E \text{ from both sides:} \quad && 0 &\geq 6 - 3V + E \\
\text{Add } 3V \text{ to both sides:} \quad && 3V &\geq 6 + E \\
\text{Subtract 6 from both sides:} \quad && 3V - 6 &\geq E
\end{aligned}$$

Theorem

If a simple connected graph is planar, then $3V - 6 \geq E$.

Thus $2E = W$ and $W \geq 3F$, so

$2E \geq 3F$.

By Euler's Formula, $V - E + F = 2$ so $F = 2 - V + E$.

Thus	$2E \geq 3F$
so	$2E \geq 3(2 - V + E)$
so	$2E \geq 6 - 3V + 3E$
Subtract $2E$ from both sides:	$0 \geq 6 - 3V + E$
Add $3V$ to both sides:	$3V \geq 6 + E$
Subtract 6 from both sides:	$3V - 6 \geq E$

Theorem

If a simple connected graph is planar, then $3V - 6 \geq E$.

Thus $2E = W$ and $W \geq 3F$, so

$2E \geq 3F$.

By Euler's Formula, $V - E + F = 2$ so $F = 2 - V + E$.

$$
\begin{array}{rr}
\text{Thus} & 2E \geq 3F \\
\text{so} & 2E \geq 3(2 - V + E) \\
\text{so} & 2E \geq 6 - 3V + 3E \\
\text{Subtract } 2E \text{ from both sides:} & 0 \geq 6 - 3V + E \\
\text{Add } 3V \text{ to both sides:} & 3V \geq 6 + E \\
\text{Subtract 6 from both sides:} & 3V - 6 \geq E
\end{array}
$$

Theorem

If a simple connected graph is planar, then $3V - 6 \geq E$.

Thus $2E = W$ and $W \geq 3F$, so

$2E \geq 3F$.

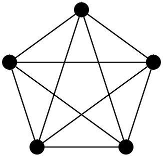
By Euler's Formula, $V - E + F = 2$ so $F = 2 - V + E$.

$$\begin{aligned}
\text{Thus} \quad && 2E &\geq 3F \\
\text{so} \quad && 2E &\geq 3(2 - V + E) \\
\text{so} \quad && 2E &\geq 6 - 3V + 3E \\
\text{Subtract } 2E \text{ from both sides:} \quad && 0 &\geq 6 - 3V + E \\
\text{Add } 3V \text{ to both sides:} \quad && 3V &\geq 6 + E \\
\text{Subtract } 6 \text{ from both sides:} \quad && 3V - 6 &\geq E
\end{aligned}$$

Theorem

If a simple connected graph is planar, then $3V - 6 \geq E$.

Conclusion



Corollary

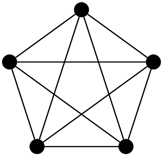
The complete graph K_5 is not planar.

Proof.

- We know that K_5 has 5 vertices and 10 edges.
- If K_5 actually were planar, then by our theorem, $3V - 6 \geq E$.
- But $3 \cdot 5 - 6 = 15 - 6 = 9$, which is $\not\geq 10$.
- So this is impossible! Thus K_5 is not planar.

$\square$

Conclusion



Corollary

The complete graph K_5 is not planar.

Proof.

- We know that K_5 has 5 vertices and 10 edges.
- If K_5 actually were planar, then by our theorem, $3V - 6 \geq E$.
- But $3 \cdot 5 - 6 = 15 - 6 = 9$, which is $\not\geq 10$.
- So this is impossible! Thus K_5 is not planar.

$\square$

Warning

Theorem

If a graph is planar, then $3V - 6 \geq E$.

Idea

If $3V - 6 \geq E$, then the graph is planar.

Warning

Theorem

If a graph is planar, then $3V - 6 \geq E$.

Idea

If $3V - 6 \geq E$, then the graph is planar.

Warning

Theorem

If a graph is planar, then $3V - 6 \geq E$.

FALSE!

Idea

If $3V - 6 \geq E$, then the graph is planar.

Warning

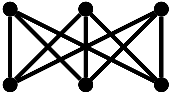
Theorem

If a graph is planar, then $3V - 6 \geq E$.

FALSE!

Idea

~~If $3V - 6 \geq E$, then the graph is planar.~~



$V = 6$ and $E = 9$, so $3V - 6 = 12 \geq 9$,
but this graph is *not* planar!