16.9 The Divergence Theorem

- Motivation:

- Divergence of a vector function:

1. Calculate the divergence of $\mathbf{F} = \langle e^x \sin y,\ e^x \cos y,\ z \rangle$.

- Divergence Theorem: Let E be a simple solid region in $\mathbb{R}^3$ and $\mathcal{S} = \partial E$ be the boundary surface of E with outward orientation. Let $\mathbf{F}$ be a vector field whose components have continuous first-order partial derivatives. Then

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\mathbf{F})\, dV.$$

- Discussion:

2. Verify the Divergence Theorem for the vector field $\mathbf{F}(x, y, z) = \langle 2x,\ -yz,\ z^2 \rangle$ over the closed region E, the solid bounded by the paraboloid $z = x^2 + y^2$ capped by the disk $x^2 + y^2 \leq 1$ in the plane $z = 1$.

3. Use the Divergence Theorem to calculate the flux of the vector field $\mathbf{F} = \langle -x^3,\ -y^3,\ 3z^2 \rangle$ across the boundary $\mathcal{S}$ of

$$E = \left\{ (x, y, z) :\ x^2 + y^2 < 16,\ 0 < z < 5 \right\}.$$

4. Evaluate the surface integral $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F} = \left\langle x^2 - z^2,\ e^{z^2} - \cos x,\ y^3 \right\rangle$$

and $\mathcal{S}$ is the boundary of the region E bounded by

$$x + 2y + 4z = 12$$

and the coordinate planes in the first octant.