

## 16.4 Green's Theorem

- Circulation—a line integral of a vector field around a closed curve. What is the circulation of a gradient vector field?
- Suppose  $\mathcal{C}$  is a simple closed curve that forms the boundary of a region  $\mathcal{D}$  in the  $xy$ -plane. Let  $\mathcal{C}$  be oriented so that traversing  $\mathcal{C}$  in the positive direction keeps  $\mathcal{D}$  to the left, i.e. counterclockwise orientation. The corresponding line integral of  $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$  around  $\mathcal{C}$  is denoted by  $\oint_{\mathcal{C}} P(x, y) \, dx + Q(x, y) \, dy$ .
- Green's Theorem: Let  $\mathcal{D}$  be a domain whose boundary  $\mathcal{C} = \partial\mathcal{D}$  is a simple closed piecewise smooth curve in the plane, positively oriented (counterclockwise). If  $\mathbf{F} = \langle P, Q \rangle$ , where  $P(x, y)$  and  $Q(x, y)$  are continuous and have continuous partial derivatives, then

1. Evaluate  $\oint_{\mathcal{C}} x^2 y \, dx + x \, dy$  using Green's Theorem, and then check the answer by evaluating the line integral directly, where  $\mathcal{C}$  is the triangle connecting the points  $(0, 0)$ ,  $(1, 0)$ , and  $(1, 2)$ , oriented counterclockwise.

2. Find the work done by the force field  $\mathbf{F} = \langle e^x - y^3, \cos y + x^3 \rangle$  on a particle traveling around the unit circle, counterclockwise.

3. Area:

4. Use a line integral to find the area of the region enclosed by the astroid

$$x = a \cos^3 t, \quad y = a \sin^3 t, \quad 0 \leq t \leq 2\pi.$$

5. Use Green's Theorem to evaluate  $\oint_{\mathcal{C}} \ln(1 + y) \, dx - \frac{xy}{1 + y} \, dy$ , where  $\mathcal{C}$  is the triangle with vertices  $(0, 0)$ ,  $(2, 0)$ , and  $(0, 4)$ , positively oriented.

6. Find the line integral of  $\mathbf{F} = \langle x^2 + y, 4x - \cos y \rangle$  around the boundary of the region  $\mathcal{D}$  that is inside the square with vertices  $(0, 0)$ ,  $(5, 0)$ ,  $(5, 5)$ , and  $(0, 5)$ , but is outside the rectangle with vertices  $(1, 1)$ ,  $(3, 1)$ ,  $(3, 2)$ , and  $(1, 2)$ . Assume the boundary of  $\mathcal{D}$  is oriented such that  $\mathcal{D}$  is on the left when the boundary is traversed.