16.3 Fundamental Theorem for Line Integrals

1. Let $\boldsymbol{f(x,y) = xy^3 + x^2}$. Compute the work done along the curve $\boldsymbol{\mathcal{C}}$ by the gradient vector field of $\boldsymbol{f}$ (see picture).

2. **Fundamental Theorem for Line Integrals**

3. Note: For a closed curve $\boldsymbol{\mathcal{C}}$, $\mathbf{F} = \boldsymbol{\nabla f}$ implies that $\displaystyle\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \mathbf{0}$.

4. Evaluate $\displaystyle\int_{\mathcal{C}} \boldsymbol{\nabla f} \cdot d\mathbf{r}$, where $\boldsymbol{f(x,y,z) = \cos(\pi x) + \sin(\pi y) - xyz}$ for any curve $\boldsymbol{\mathcal{C}}$ from $\boldsymbol{(1, \frac{1}{2}, 2)}$ to $\boldsymbol{(2, 1, -1)}$.

5. **Equivalent Conditions for Path Independence**

Assume $\mathbf{F}$ is a continuous vector field in some domain D.

- $\mathbf{F}$ is a conservative vector field if there is a function $\boldsymbol{f}$ such that $\mathbf{F} = \boldsymbol{\nabla f}$. The function $\boldsymbol{f}$ is called a potential function for the vector field.
- $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path if $\displaystyle\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths $\boldsymbol{\mathcal{C}_1}$ and $\boldsymbol{\mathcal{C}_2}$ in D with the same initial and final points.
- A path $\boldsymbol{\mathcal{C}}$ is closed if its initial and final points coincide. For example, any circle is a closed path.
- A path $\boldsymbol{\mathcal{C}}$ is simple if it does not cross itself. For example, a circle is a simple curve, but a figure 8 is not simple.
- A region D is open if it does not contain any of its boundary points.
- A region D is connected if we can connect any two points in the region with a path that lies completely in D.
- A region D is simply connected if it is connected and it contains no holes.

6. Conservative Vector Fields: Given a vector field, when is it conservative? If it is conservative, what is the potential?

7. Existence of a Potential Function: Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field on a simply-connected domain D. If the cross partials of $\mathbf{F}$ are equal, that is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

then $\mathbf{F} = \boldsymbol{\nabla f}$ for some potential $\boldsymbol{f}$, and $\mathbf{F}$ is a conservative vector field.

8. Determine if the following vector fields are conservative and find a potential function for the vector field if it is conservative.

   (a) $\mathbf{F} = (y + x)\mathbf{i} + (y - x)\mathbf{j}$

   (b) $\mathbf{F} = (2x - 3y)\mathbf{i} + (-3x + 4y - 8)\mathbf{j}$

9. Given $\mathbf{F}(x,y,z) = yz\,\mathbf{i} + xz\,\mathbf{j} + (xy+2z)\mathbf{k}$, find a function f such that $\mathbf{F} = \nabla f$, then evaluate $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ along the line segment C from $(1,0,-2)$ to $(4,6,3)$.

10. Given $\mathbf{F}(x,y) = xy^2\,\mathbf{i} + x^2y\,\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$, then evaluate $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ along the curve C: $\mathbf{r}(t) = \left\langle t + \sin\frac{\pi t}{2},\ t + \cos\frac{\pi t}{2} \right\rangle$ for $0 \le t \le 1$.

11. Vortex Vector Field: Consider $\mathbf{F} = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$; please see the vector plots: