# 14.6 The Gradient Vector and Directional Derivatives

- Gradient Vector: If $f$ is a function of two variables $x$ and $y$, then the gradient of $f$ is the vector function $\boldsymbol{\nabla f}$ defined by

1. Find the gradient of $f(x,y) = \sin x + e^{xy}$ at $(0,1)$.

2. The air temperature at ground level $(x,y)$ is given (in Celsius) by the function

$$T(x,y) = 20 - x - y + 0.05x^3 - 0.2y^2.$$

Find the temperature gradient at the origin and at the point $(-4,1)$. (See picture.)

- Recall for $z = f(x,y)$ we defined the partial derivatives at $(a,b)$ by

- Directional Derivative: The directional derivative of $f$ at $(a, b)$ in the direction of a unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ is

  if this limit exists. If $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$, then $D_{\mathbf{i}} f(a, b) = f_x(a, b)$, and if $\mathbf{u} = \mathbf{j} = \langle 0, 1 \rangle$, then $D_{\mathbf{j}} f(a, b) = f_y(a, b)$.

- Theorem: If $f$ is a differentiable function of $x$ and $y$, then $f$ has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ and

  Using the gradient vector we may rewrite the directional derivative in the direction of the unit vector $\mathbf{u}$ as

3. Find the directional derivative of $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

4. Find the directional derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

5. Find the directional derivative of $f(x, y) = x^3 - 3xy + 4y^2$ in the direction $\theta = \pi/6$, that is, in the direction that points upward and to the right at an angle of $\theta$. What is $D_{\mathbf{u}} f(1, 2)$?

- Three Variables: The gradient for $f(x, y, z)$ is

  and the directional derivative in the direction of a unit vector $\mathbf{u}$ is

- Recall the angle between two vectors.

- Theorem: Suppose that $f$ is a differentiable function of two (or three) variables. Then

  (a) the maximum value of the directional derivative $D_{\mathbf{u}}f(x, y)$ (the maximum rate of change of $f$) at $(a, b)$ is $|\boldsymbol{\nabla f(a, b)}|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\boldsymbol{\nabla f(x, y)}$: $f$ increases most rapidly in the direction of the gradient;

  (b) the minimum value of the directional derivative $D_{\mathbf{u}}f(x, y)$ (the minimum rate of change of $f$) at $(a, b)$ is $-|\boldsymbol{\nabla f(a, b)}|$ and it occurs when $\mathbf{u}$ has the opposite direction as the gradient vector $\boldsymbol{\nabla f(x, y)}$: $f$ decreases most rapidly in the direction opposite the gradient;

  (c) the rate of change of $f$ at $(a, b)$ is 0 in the directions orthogonal to $\boldsymbol{\nabla f(a, b)}$;

  (d) the gradient $\boldsymbol{\nabla f(a, b)}$ is orthogonal to the level curve $f(x, y) = c$ at the point $(a, b)$, where $c = f(a, b)$.

  Proof:

6. Find a unit vector in the direction in which $f(x, y) = 20 - x^2 - y^2$ decreases most rapidly at $P(-1, -3)$, and find the rate of change of $f$ at $P$ in that direction. At what angle does the path in that direction begin?

7. If $f(x, y) = xe^y$, find the rate of change of $f$ at the point $P(2, 0)$ in the direction from $P$ to $Q(1/2, 2)$. In what direction does $f$ have the maximum rate of change? What is this maximum rate of change?

8. Find the direction in which the function $f(x, y) = e^{2x+3y-1}$ increases most rapidly, starting at the point $(3, 7)$, and calculate its directional derivative in that direction.

9. If the elevation of a hill is given by $f(x, y) = 200 - y^2 - 4x^2$, in which direction will rain water run off the hill at the site $(1, 2)$?