14.4 Tangent Planes and Linear Approximations

- Tangent Plane:

1. Find the tangent plane to the surface $z = 3y^2 - 2x^2 + x$ at the point $(2, -1, -3)$.

2. Find the tangent plane to the surface $z = \sqrt{4 - x^2 - 2y^2}$ at the point $(1, -1, 1)$.

- Linearization/Linear Approximations:

3. Find the linearization at $(0, 0)$ of the function $f(x, y) = 1 + y + x \cos y$.

4. If $f(x, y) = x^2 + y^2$ and the local linearization to f at a point P is given by

$$L(x, y) = 2y - 2x - 2,$$

find P.

- Recall that for a function of one variable, $y = f(x)$, if x changes from a to $a + \Delta x$, we defined the increment of y as

$$\Delta y = f(a + \Delta x) - f(a).$$

If f is differentiable at a, then

$$\Delta y = f'(a)\Delta x + \varepsilon \Delta x,$$

where $\varepsilon \to 0$ as $\Delta x \to 0$.

- Increment of z: If $z = f(x, y)$ and (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$, then the increment of z is given by

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b).$$

- Differentiable: If $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

- Theorem: If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

5. Show that $f(x, y) = \dfrac{x}{x+y}$ is differentiable at $(2, 1)$ and find its linearization there.

6. If $f(x, y)$ is differentiable at $(3, 4)$ with $f(3, 4) = 5$, $f_x(3, 4) = 2$ and $f_y(3, 4) = -1$, estimate the value of $f(3.01, 3.98)$.

- Differentials: For a differentiable function $z = f(x, y)$ we define the differential dz, also called the total differential, to be

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy,$$

where the differentials dx and dy are independent variables.

If $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ then the differential of z is

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

7. If $z = f(x, y) = x^2 + 2xy - 4x$, find the differential dz. If (x, y) changes from $(1, 2)$ to $(1.01, 2.04)$, compare the values of Δz and dz.

8. The length and width of a rectangle are measured as 30 cm and 24 cm, respectively, with an error in measurement of at most 0.1 cm in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.