14.3 Partial Derivatives

• Let f(x, y) be a function of two variables, where y = b is fixed. Then g(x) = f(x, b) is a function of a single variable x. If g has a derivative at a, then we call it the partial derivative of f with respect to x at (a, b) and write

$$f_x(a,b) = g'(a).$$

• Now keep x = a fixed, and let h(y) = f(a, y). If h has a derivative at b, then we call it the partial derivative of f with respect to y at (a, b) and write

$$f_y(a,b) = h'(b).$$

• By the definition of a derivative, we have

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h},$$

 $f_y(a,b) = \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k}.$

The partial derivatives of f(x, y) are the functions $f_x(x, y)$ and $f_y(x, y)$ obtained by letting the point (a, b) vary.

• Notation: If z = f(x, y), then we may write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f,$$

 $f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f.$

- To find f_x regard y as a constant and differentiate f(x,y) with respect to x. To find f_y regard x as a constant and differentiate f(x,y) with respect to y.
- 1. Find all first partial derivatives for $f(u,v)=(u^2v-v^3)^5$ and $g(s,t)=\tan^{-1}(st^2)$.

2. If $f(x,y) = x^2 + 3x^3y - xy^2$ find $f_x(0,1)$ and $f_y(1,0)$.

3. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the following functions.

$$f(x,y) = \frac{x^y}{\cos x + \sin y}$$

$$f(x,y) = e^{ax^2 + by^2 + c}$$

$$f(x,y) = \ln(x^4 + 2y^3)$$

4. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation

$$3x^3 + 2y^3 + z^3 + 6xyz = 1.$$

- 5. Interpretations: Partial derivative can be interpreted as rates of change. The geometric interpretation: the partial derivatives are the slopes of the tangent lines at P(a, b, c) to the curves given by the intersection of the surface given by z = f(x, y) and the planes x = a and y = b.
- 6. If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables.
- 7. Second Partials: The second partial derivatives of f are

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

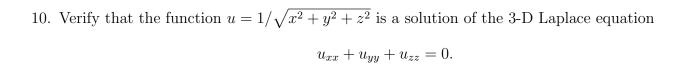
$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}.$$

8. Find all second partial derivatives of

$$f(x,y) = x^3 + x^2y^3 - 2y^2$$

9. Clairaut's Theorem: Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b).$$



11. Find the partial derivatives of
$$f(x,y) = \int_{x}^{y} e^{t^2+t+1} dt$$
.

12. Find
$$f_x$$
, f_y , f_{xy} , f_{yx} for $f(x, y) = xye^{3xy}$.