14.2 Limits and Continuity

- Limit Definition: For a function f of two variables whose domain D contains points arbitrarily close to (a, b),
$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$
if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that $|f(x, y) - L| < \varepsilon$ whenever $(x, y) \in D$ and
$$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta.$$

- Important Note: If
$$\lim_{(x,y)\to(a,b)} f(x, y) = L_1 \quad \text{along a path } C_1$$
but we calculate that
$$\lim_{(x,y)\to(a,b)} f(x, y) = L_2 \neq L_1 \quad \text{along another path } C_2$$
then the overall limit $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

1. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}$ does not exist.

2. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

3. Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2 \sin^2 x}{x^4 + y^4}$ exist?

4. Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y e^y}{x^4 + 4y^2}$ exist?

5. Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$ if it exists.

6. Determine $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$.

7. Determine $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{\sqrt{x^2+y^2}}$.

- Continuity: A function f of two variables is continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

  Examples: polynomials, rational functions, trigonometric functions, exponential functions, and logarithmic functions are all continuous on their domains.

8. Determine the set of points at which the function is continuous: $F(x,y) = \dfrac{xy}{1+e^{x-y}}$.

9. Determine the set of points at which the function is continuous: $G(x,y) = \ln(x^2+2y^2-4)$.

10. Determine the largest set on which the function $f(x,y) = \dfrac{2xy}{9-x^2-y^2}$ is continuous.

11. Is f continuous? $f(x,y) = \begin{cases} \dfrac{x^2y^3}{2x^2+y^2} & :(x,y)\neq(0,0), \\ 1 & :(x,y)=(0,0). \end{cases}$