13.3b Curvature

1. Curvature I: The curvature κ (kappa) of a curve is the scalar quantity $\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$, the magnitude of the rate of change of the unit tangent vector $\mathbf{T}$ with respect to arc length s along the curve. This can be rewritten as
$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}, \quad \text{where} \quad |\mathbf{r}'(t)| \neq 0.$$

2. Find the curvature of the circle of radius q centered at (a, b).

3. Curvature II: The curvature κ of a curve is also given by
$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3}.$$

4. Find the curvature of $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + (1 - t^2)\mathbf{k}$. When is the maximum curvature?

5. Curvature for a plane curve $y = f(x)$: For the plane curve $y = f(x)$ we have $\mathbf{r}(t) = \langle t, f(t), 0\rangle$, and the curvature is given by

$$\kappa = \frac{|f''(x)|}{[f'(x)^2 + 1]^{3/2}}.$$

6. Unit Normal Vector: Since the unit tangent vector $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ has constant norm (of 1), $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are orthogonal. Thus the unit normal vector is a unit vector in the same direction as $\mathbf{T}'(t)$ given by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}.$$

13.3b Curvature

For practice: (860) 17,23,25,33,39,47,51a

ASSIGNMENT TO BE TURNED IN:

1. Find, in terms of β, the curvature of the helix $\mathbf{r}(t) = \langle \sin t, \cos t, \beta t \rangle$, where β is some constant.

2. Calculate $\mathbf{r}'(t)$ and $\mathbf{T}(t)$, and evaluate $\mathbf{T}(1)$ for $\mathbf{r}(t) = \langle 1+2t, t^2, 3-t^2 \rangle$.

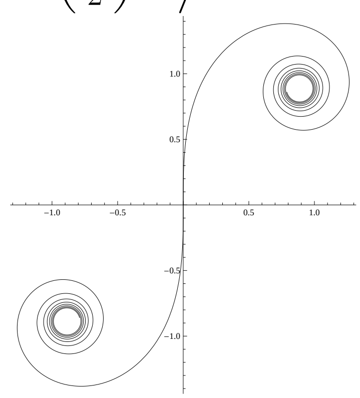
3. Find the curvature of the plane curve $y = t^n$ at the point $t = 1$. Your answer will involve n.

4. (a) Show that the curvature function of the parametrization $\mathbf{r}(t) = \langle a\cos t, b\sin t \rangle$ of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ is

$$\kappa(t) = \frac{ab}{\left(b^2\cos^2 t + a^2\sin^2 t\right)^{3/2}}.$$

   (b) Use this equation for $\kappa(t)$ to find the t values at which the maximum and minimum curvature occurs on the ellipse, assuming $b > a$. Lastly, what happens to the curvature if $a = b$?

5. The Cornu spiral is the plane curve $\mathbf{r}(t) = \left\langle \int_0^t \sin\left(\frac{u^2}{2}\right) du, \int_0^t \cos\left(\frac{u^2}{2}\right) du \right\rangle$. Find $\kappa(t)$ for



the Cornu spiral; your answer should have an absolute value in it.

6. Find the unit normal vector to the Cornu spiral (previous problem) at $t = \sqrt{\pi}$.