13.3a Arc Length and Speed

For practice: (860) 1,11,13,15

Example: Find the arc length parametrization of $\mathbf{r}(t) = 2t\mathbf{i} + (1-2t)\mathbf{j} + t\mathbf{k}$: Since

$$|\mathbf{r}'(t)| = |\langle 2, -2, 1\rangle| = 3,$$

the arc length function is

$$s(t) = \int_0^t |\mathbf{r}'(t)|dt = 3t.$$

Solving for t in terms of s, $t = \varphi(s) = s/3$. Then an arc length parametrization is

$$\mathbf{r}_1(s) = \mathbf{r}(\varphi(s)) = \mathbf{r}(s/3) = \left\langle \frac{2s}{3}, 1 - \frac{2s}{3}, \frac{s}{3} \right\rangle.$$

Note that $\mathbf{r}_1'(s) = \langle 2/3, -2/3, 1/3\rangle$, and $|\mathbf{r}_1'(s)| = 1$.

ASSIGNMENT TO BE TURNED IN:

1. Find the length of the curve $\mathbf{r}(t) = (2t^2+1)\mathbf{i} + (2t^2-1)\mathbf{j} + t^3\mathbf{k}$ for $0 \le t \le 4$.
2. Find the speed of $\mathbf{r}(t) = \sin 3t\mathbf{i} + \cos 4t\mathbf{j} + \cos 5t\mathbf{k}$ at $t = \frac{\pi}{2}$.
3. Which of the following is an arc length parametrization of a circle of radius 4 centered at the origin? (Yes or No, show work.)
   (a) $\mathbf{r}_1(t) = 4\sin t\mathbf{i} + 4\cos t\mathbf{j}$
   (b) $\mathbf{r}_2(t) = 4\sin 4t\mathbf{i} + 4\cos 4t\mathbf{j}$
   (c) $\mathbf{r}_3(t) = 4\sin \frac{t}{4}\mathbf{i} + 4\cos \frac{t}{4}\mathbf{j}$
4. Find an arc length parametrization of the circle in the plane $z = 9$ with radius 4 and center $(1, 4, 9)$.
5. Show that one arch of the cycloid $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$ has length 8. Find the value of t in $[0, 2\pi]$ where the speed is at a maximum. Hint: You'll need a half-angle formula.
6. Find an arc length parametrization of $\mathbf{r}(t) = \langle t^2, t^3\rangle$.
7. A helix of radius R and height h making N complete turns has the parametrization

$$\left\langle R\cos\left(\frac{2\pi N t}{h}\right), R\sin\left(\frac{2\pi N t}{h}\right), t\right\rangle$$

   for $0 \le t \le h$. Use this to find a general formula for the length of a helix of radius R and height h that makes N complete turns.