12.4 The Cross Product

1. 2×2 Determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

2. 3×3 Determinant: $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$

3. Compute $\begin{vmatrix} -2 & -5 \\ 1 & 3 \end{vmatrix}$ and $\begin{vmatrix} 2 & 3 & -1 \\ 0 & 1 & 0 \\ -2 & -1 & 3 \end{vmatrix}$.

4. Cross Product: The cross product of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ in $\mathbb{R}^3$ is the vector
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}.$$

5. Compute $\langle 1, -3, 5 \rangle \times \langle -2, 4, 6 \rangle$.

6. Theorem: For any vector $\mathbf{a} \in \mathbb{R}^3$ we have $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ and $\mathbf{a} \times \mathbf{0} = \mathbf{0}$.

7. Theorem: For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ the cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal (normal) to both $\mathbf{a}$ and $\mathbf{b}$.

8. Right-hand Rule: If you align the fingers of your right hand along the vector $\mathbf{a}$ and bend your fingers around in the direction of rotation from $\mathbf{a}$ toward $\mathbf{b}$ through an angle of less than $180°$, your thumb will point in the direction of $\mathbf{a} \times \mathbf{b}$. Note that $\mathbf{b} \times \mathbf{a}$ points in the opposite direction of $\mathbf{a} \times \mathbf{b}$.

9. Warning! $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$, and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ in general.

10. Theorem: For any vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ and any scalar λ, the following hold:

   (a) $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$ Anticommutativity

   (b) $(\lambda\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\lambda\mathbf{b}) = \lambda(\mathbf{a} \times \mathbf{b})$

   (c) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ Distributive law

   (d) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$ Distributive law

   (e) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ Scalar triple product

   (f) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ Vector triple product

11. Theorem: For nonzero vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, if θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and $0 \leq \theta \leq \pi$, then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\ |\mathbf{b}|\ \sin\theta.$$

12. Corollary: Two nonzero vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \vec{0}$.

13. Find $|\mathbf{a} \times \mathbf{b}|$ and determine whether $\mathbf{a} \times \mathbf{b}$ is directed into the page or out of the page.

14. Area of Parallelogram: The parallelogram formed by two nonparallel nonzero vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ has an area given by the norm of their cross product: Area $= |\mathbf{a} \times \mathbf{b}|$.

15. Compute the area of the parallelogram formed by vectors $\langle 1, 2, 3\rangle$ and $\langle 4, 5, 6\rangle$.

16. Note: $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

17. Volume of Parallelepiped and Scalar Triple Product: The parallelepiped formed by three non-coplanar vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ has a volume given by: Volume $= |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

18. Compute the volume of the parallelepiped with three adjacent edges formed by the vectors $\mathbf{a} = \langle 1, 2, 1\rangle$, $\mathbf{b} = \langle 2, 1, -3\rangle$ and $\mathbf{c} = \langle 2, -1, 3\rangle$.

19. Torque $\vec{\boldsymbol{\tau}}$: A force known as torque $\vec{\boldsymbol{\tau}}$ is defined by $\vec{\boldsymbol{\tau}} = \mathbf{r} \times \mathbf{F}$, where $\mathbf{F}$ is the force applied to the end of a handle and $\mathbf{r}$ is the position vector for the end of the handle. In particular,

$$|\vec{\boldsymbol{\tau}}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}|\ |\mathbf{F}|\ \sin\theta.$$

20. Compute the magnitude of the torque if a 40-newton force is applied to the end of a 0.25-meter wrench at an angle of 75°. Give your answer in joules, where 1 joule is 1 newton-meter.

21. Point-to-Line Distance in $\mathbb{R}^3$: The distance d from a point Q to the line through the points P and R in $\mathbb{R}^3$ is

$$d =$$

22. Compute the distance from $Q(1, 2, 1)$ to the line through the points $P(2, 1, -3)$ and $R(2, -1, 3)$.