12.3 The Dot Product

1. Dot Product: The dot product of two vectors $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ in $\mathbb{R}^2$ is the scalar

   $\mathbf{a} \cdot \mathbf{b} =$

   Similarly, the dot product of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ in $\mathbb{R}^3$ is

   $\mathbf{a} \cdot \mathbf{b} =$

2. Compute the dot product $\mathbf{a} \cdot \mathbf{b}$ for the following:

   (a) $\mathbf{a} = \langle 3, -2, -4 \rangle$ and $\mathbf{b} = \langle 2, 1, 5 \rangle$;

   (b) $\mathbf{a} = 2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$.

3. Theorem: For vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ and any scalar λ the following hold:

   (a) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ Commutative law

   (b) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ Distributive law

   (c) $(\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (\lambda \mathbf{b})$

   (d) $\mathbf{0} \cdot \mathbf{b} = 0$

   (e) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$.

   Proof:

4. Angle between vectors: For two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, define the angle θ, $0 \leq \theta \leq \pi$, to be the smaller angle between the vectors.

5. Theorem: Let θ be the angle between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$. Then

   $$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \cos\theta.$$

   Proof:

6. Compute the angle between the vectors $\mathbf{a} = \langle 2, 3, 5 \rangle$ and $\mathbf{b} = \langle -4, 1, -1 \rangle$.

7. Orthogonal: Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal iff $\mathbf{a} \cdot \mathbf{b} = 0$.
8. Find all values of λ such that $\mathbf{a} = \langle \lambda, -2, 3 \rangle$ and $\mathbf{b} = \langle \lambda, \lambda, -5 \rangle$ are orthogonal.
9. The angle between $\mathbf{a}$ and $\mathbf{b}$ is acute if $\mathbf{a} \cdot \mathbf{b} > 0$ and obtuse if $\mathbf{a} \cdot \mathbf{b} < 0$.
10. Find a vector orthogonal to $4\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.
11. Component of $\mathbf{b}$ along $\mathbf{a}$: The component of $\mathbf{b}$ along $\mathbf{a}$ is the scalar

    $$\text{comp}_{\mathbf{a}}\mathbf{b} =$$

12. Projection of $\mathbf{b}$ onto $\mathbf{a}$: The projection of $\mathbf{b}$ onto $\mathbf{a}$ is the vector

    $$\text{proj}_{\mathbf{a}}\mathbf{b} =$$

13. Compute $\text{comp}_{\mathbf{b}}\mathbf{a}$ and $\text{proj}_{\mathbf{b}}\mathbf{a}$ for $\mathbf{a} = \langle 2, -1, 3 \rangle$ and $\mathbf{b} = \langle 1, 2, 2 \rangle$.
14. Prove that $\mathbf{a} \cdot \mathbf{b} = \frac{1}{4}|\mathbf{a} + \mathbf{b}|^2 - \frac{1}{4}|\mathbf{a} - \mathbf{b}|^2$.