Upper and Lower Semimodularity of the Supercharacter Theory Lattices of Cyclic Groups

Samuel Benidt, William Hall, & Anders Hendrickson

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Outline

1 Definitions
   • Lattices
   • Supercharacter Theories

2 Our Work
   • Goals and Strategy
   • Analysis of Specific Cases

3 Two main theorems
   • Upper Semimodularity
   • Lower Semimodularity

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Modularity Conditions of Lattices of Cyclic Groups
As you may know, a partially ordered set is a set with an order that requires the following for elements $a, b$ in the set:

- either $a \leq b$
- or $a \geq b$
- or $a$ and $b$ are incomparable

A good example of a poset is set inclusion: either $A \subseteq B$, or $B \subseteq A$, or neither.

**Definition**

A **lattice** is a partially ordered set in which any two elements $a$ and $b$ have a unique least upper bound, $a \lor b$, and greatest lower bound, $a \land b$. We read $a \lor b$ as “$a$ join $b$” and $a \land b$ as “$a$ meet $b$”.
Definitions

We say an element $b$ of a lattice covers another element $d$ if $b \geq d$ and there are no elements between $b$ and $d$. We write $d \prec b$. 

A Lattice
Examples

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A Lattice

Not a lattice
Definitions

Two main theorems

Lattices
Supercharacter Theories

Examples

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Definitions

A lattice $L$ of finite length, is said to be upper semimodular if the following condition is satisfied:

$$
\text{if } a \land b \preceq a, b \text{ then } a, b \preceq a \lor b.
$$
A lattice $L$ of finite length, is said to be upper semimodular if the following condition is satisfied:

$$a \land b \rightarrow a, b \text{ then } a, b \rightarrow a \lor b.$$
Examples of some USM Lattices
**Definition**

Let $L$ be a lattice of finite length. Then $L$ is lower semimodular if for all $a, b \in L$,

$$
\text{if } a, b \triangleleft a \lor b \text{ then } a \land b \triangleleft a, b
$$

![Diagram](image)
Definition

Let $L$ be a lattice of finite length. Then $L$ is lower semimodular if for all $a, b \in L$,

if $a, b \preceq a \lor b$ then $a \land b \preceq a, b$
Examples of some LSM Lattices
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Examples of some LSM Lattices
In our presentation, we’ll be focusing exclusively on finite cyclic groups and their subgroups.

Note that since every group we’ll be discussing is Abelian, each subgroup $N$ is normal.

This means that the quotient group $G/N$ exists.
Supercharacter Theory

Definition

Let $G$ be a finite group and let $K$ be a partition of $G$. Then we say $K$ is a supercharacter theory if the following three conditions hold:

- $\{1\}$ is a part of $K$.
- A product of sums of parts of $K$ is a linear combination of sums of parts of $K$.
- Each part of $K$ is a union of some conjugacy classes of $G$.

$\text{Sup}(G)$ denotes the set of all supercharacter theories of the group $G$; it forms a lattice.

- Supercharacter theories were defined in a 2008 paper by P. Diaconis and I.M. Isaacs.
- They are the subject of very active study and the topic of an American Institute of Mathematics workshop this May.
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Sup(G) as a Lattice

Given $X, Y \in \text{Sup}(G)$, we can find their join and meet, so Sup(G) is a lattice.
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Given \( X, Y \in \text{Sup}(G) \), we can find their join and meet, so \( \text{Sup}(G) \) is a lattice.

\[
\begin{align*}
X \;
\end{align*}
\]

\[
\begin{align*}
Y \;
\end{align*}
\]

\[
\begin{align*}
X \vee Y = a_1, a_2, a_3, a_4 \\
&
\end{align*}
\]

\[
\begin{align*}
X \wedge Y = a_1, a_2, a_3, a_4 \\
&
\end{align*}
\]
Sup(G) as a Lattice

Given $X, Y \in \text{Sup}(G)$, we can find their join and meet, so $\text{Sup}(G)$ is a lattice.
Some Examples: $\text{Sup}(\mathbb{Z}_3)$

Figure: The graph of $\text{Sup}(\mathbb{Z}_3)$
Some Examples: $\text{Sup}(\mathbb{Z}_{17})$

Figure: The graph of $\text{Sup}(\mathbb{Z}_{17})$
Some Examples: $\text{Sup}(\mathbb{Z}_{51})$

**Figure:** The graph of $\text{Sup}(\mathbb{Z}_{51})$
There are several methods of constructing supercharacter theories of a group, including the following:

- **minimal supercharacter theory**: $m(\mathbb{Z}_n)$
  $m(\mathbb{Z}_n)$ is formed by placing each group element in its own part. This partition will always be the least element in the lattice.

- **maximal supercharacter theory**: $M(\mathbb{Z}_n)$
  $M(\mathbb{Z}_n)$ is formed by placing all group elements in two parts: \{1\}, and all other group elements. This partition will always be the greatest element in the lattice.

- **inverse supercharacter theory**: $\text{Inv}(\mathbb{Z}_n) = \{1/g, g^{-1}/g^2, g^{-2}/\ldots\}$
  We define the inverse supercharacter theory by placing each group element and its inverse together in their own part.

- **partition the elements by their orders**: $A(\mathbb{Z}_n)$

- ***-product supercharacter theory**: $X * Y$
  where $X \in \text{Sup}(N)$ and $Y \in \text{Sup}(G/N)$. 
As mentioned on the previous slide, one way to find supercharacter theories is a method called the $\ast$-product.

Let $N$ be normal in $G$ with $X \in \text{Sup}(N)$, $Y \in \text{Sup}(G/N)$. Then $X \ast Y \in \text{Sup}(G)$.
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\[ X \ast Y \]
\*-product

As mentioned on the previous slide, one way to find supercharacter theories is a method called the \*-product.

Let $N$ be normal in $G$ with $X \in \text{Sup}(N)$, $Y \in \text{Sup}(G/N)$. Then $X \ast Y \in \text{Sup}(G)$.
We seek necessary and sufficient conditions for when $\text{Sup}(G)$ is upper- and lower-semimodular.
Inheritance Lemma

Lemma (Inheritance)

Let $N$ be a subgroup of cyclic group $G$. Then

1. If $\text{Sup}(N)$ is not upper semimodular, then $\text{Sup}(G)$ is not upper semimodular.

2. If $\text{Sup}(N)$ is not lower semimodular, then $\text{Sup}(G)$ is not lower semimodular.
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∗ for odd primes
Case where $|G| = p$

**Lemma**

*Given a group $G$ of order $p$, Sup$(G)$ is isomorphic to the subgroup lattice of the automorphism group of $G$.*

**Lemma**

*It is a well-known fact in lattice theory that the subgroup lattice of an Abelian group is both USM and LSM.*

Therefore, Sup$(G)$ is both USM and LSM.
Case where $|G| = p$

**Lemma**

*Given a group $G$ of order $p$, $\text{Sup}(G)$ is isomorphic to the subgroup lattice of the automorphism group of $G$.***

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Therefore, $\text{Sup}(G)$ is both USM and LSM.
Example of $|G| = p$

Lattice of $\text{Sup}(\mathbb{Z}_{19})$

Lattice of subgroups of $\text{Aut}(\mathbb{Z}_{19})$
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\( \ast \) for odd primes

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Case where $|G| = p^2$, if $p$ is odd

If $G$ is cyclic of order $p^2$, $p$ an odd prime, then $\text{Sup}(G)$ contains the following sublattice.

Note that each line drawn in on this particular diagram indicates a covering relation, and each element listed is distinct from the others.
Case where $|G| = p^2$, if $p$ is odd

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Case where $|G| = 4$

\[ M(\mathbb{Z}_4) \rightarrow M(\mathbb{Z}_2) \ast M(\mathbb{Z}_2) \rightarrow m(\mathbb{Z}_4) \]
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*for odd primes $p$
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*for odd primes $p$
USM Case for $|G| = pq$

\[
\begin{align*}
M(\mathbb{Z}_{pq}) & \rightarrow M(\mathbb{Z}_p) \times M(\mathbb{Z}_q) & \rightarrow m(\mathbb{Z}_p) \times m(\mathbb{Z}_q) \rightarrow M(\mathbb{Z}_{pq}) \\
M(\mathbb{Z}_p) \times M(\mathbb{Z}_q) & \rightarrow m(\mathbb{Z}_p) \times m(\mathbb{Z}_q) & \rightarrow m(\mathbb{Z}_q) \times m(\mathbb{Z}_p) \rightarrow M(\mathbb{Z}_{pq}) \\
m(\mathbb{Z}_p) \times m(\mathbb{Z}_q) & \rightarrow m(\mathbb{Z}_p) \times m(\mathbb{Z}_q) & \rightarrow m(\mathbb{Z}_q) \times m(\mathbb{Z}_p) \rightarrow M(\mathbb{Z}_{pq}) \\
m(\mathbb{Z}_{pq}) & \rightarrow m(\mathbb{Z}_p) \times m(\mathbb{Z}_q) & \rightarrow m(\mathbb{Z}_q) \times m(\mathbb{Z}_p) \rightarrow M(\mathbb{Z}_{pq})
\end{align*}
\]
USM Case for $|G| = pq$

\[ M(\mathbb{Z}_{pq}) \]

\[ M(\mathbb{Z}_p) \ast M(\mathbb{Z}_q) \]

\[ m(\mathbb{Z}_p) \ast m(\mathbb{Z}_q) \]

\[ m(\mathbb{Z}_q) \ast m(\mathbb{Z}_p) \]

\[ m(\mathbb{Z}_{pq}) \]
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Lemma

Let $G$ be a cyclic group of order $pq$. Then $\text{Sup}(G)$ is lower semimodular.
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*for odd primes $p$
Case where $|G| = pqr$

\[ \mathcal{M}(\mathbb{Z}_{pqr}) \]

\[ \mathcal{M}(\mathbb{Z}_p) \ast \mathcal{M}(\mathbb{Z}_{qr}) \]

\[ \mathcal{M}(\mathbb{Z}_p) \ast \mathcal{M}(\mathbb{Z}_q) \ast \mathcal{M}(\mathbb{Z}_r) \]

\[ \text{Aut}(\mathbb{Z}_{pq}) \ast \mathcal{M}(\mathbb{Z}_r) \]

\[ \mathcal{M}(\mathbb{Z}_q) \ast \mathcal{M}(\mathbb{Z}_{pr}) \]
Case where $|G| = pqr$

\[
M(\mathbb{Z}_{pqr}) 
\rightarrow 
M(\mathbb{Z}_p) * M(\mathbb{Z}_{qr}) 
\rightarrow 
M(\mathbb{Z}_p) * M(\mathbb{Z}_q) * M(\mathbb{Z}_r) 
\rightarrow 
\text{Aut}(\mathbb{Z}_{pq}) * M(\mathbb{Z}_r) 
\rightarrow 
M(\mathbb{Z}_q) * M(\mathbb{Z}_{pr})
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*for odd primes $p$
Case where $|G| = 4p$, if $p$ is odd

\[ M(\mathbb{Z}_{4p}) \]

\[ M(\mathbb{Z}_2) \ast M(\mathbb{Z}_{2p}) \]

\[ M(\mathbb{Z}_2) \ast M(\mathbb{Z}_p) \ast M(\mathbb{Z}_2) \]

\[ A(\mathbb{Z}_{2p}) \ast M(\mathbb{Z}_2) \]

\[ M(\mathbb{Z}_p) \ast M(\mathbb{Z}_4) \]

\[ M(\mathbb{Z}_p) \ast M(\mathbb{Z}_2) \ast M(\mathbb{Z}_2) \]

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Modularity Conditions of Lattices of Cyclic Groups
Case where $|G| = 4p$, if $p$ is odd

- $M(\mathbb{Z}_{4p})$
- $M(\mathbb{Z}_2) \ast M(\mathbb{Z}_{2p})$
- $M(\mathbb{Z}_p) \ast M(\mathbb{Z}_4)$
- $M(\mathbb{Z}_2) \ast M(\mathbb{Z}_p) \ast M(\mathbb{Z}_2)$
- $M(\mathbb{Z}_p) \ast M(\mathbb{Z}_2) \ast M(\mathbb{Z}_2)$
- $A(\mathbb{Z}_{2p}) \ast M(\mathbb{Z}_2)$
Case where $|G| = 4p$, if $p$ is even

$G \cong \mathbb{Z}_8$
Case where $|G| = 4p$, if $p$ is even

$G \cong \mathbb{Z}_8$
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Upper Semimodularity Theorem

Theorem (USM)

Let \( G \) be a cyclic group. Then \( \text{Sup}(G) \) is upper semimodular if and only if the order of \( G \) is prime or four.
**USM Proof**

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**Cases where cyclic group $G$ is USM:**

- If $|G| = p$, then we proved that $\text{Sup}(G)$ is USM.
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USM Proof

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What happens for other orders of $G$?
If $|G|$ is not prime or four, then it is either a multiple of two distinct primes or a prime power.

- For $|G| = k \cdot pq$, where $p \neq q$, then we showed that $\text{Sup}(\mathbb{Z}_{pq})$ is not USM. By the Inheritance Lemma, $\text{Sup}(G)$ is not USM either.
- For $|G| = p^n$, where $p \neq 2$ and $n \geq 2$, we again showed that $\text{Sup}(\mathbb{Z}_{p^2})$ is not USM. Then, by the Inheritance Lemma, $\text{Sup}(G)$ is not USM.
- This leaves $|G| = 2^n$ with $n \geq 3$. We verified that $\text{Sup}(\mathbb{Z}_8)$ is not USM, and thus, for all higher 2-powers, $\text{Sup}(G)$ is not USM.
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Definitions

Our Work

Two main theorems

Upper Semimodularity

Lower Semimodularity

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Lower Semimodularity Theorem

Theorem (LSM)

Let $G$ be a cyclic group. Then $\text{Sup}(G)$ is lower semimodular if and only if the order of $G$ is prime, the product of two distinct primes, or four.
Cases where cyclic group $G$ is LSM:

- When $|G| = p$, $\text{Sup}(G)$ is lattice-isomorphic to the lattice $C$ of subgroups of $\text{Aut}(G)$. Since $C$ is LSM, $\text{Sup}(G)$ is also.
- When $|G| = pq$, where $p \neq q$, a previously-stated lemma proved that $\text{Sup}(G)$ is LSM.
- When $|G| = 4$, we verified that this supercharacter theory lattice is LSM.

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Two main theorems

**Upper Semimodularity**

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What happens for other orders of $G$?
Suppose $|G|$ is not $p$, $pq$, or 4. Consider the number of distinct prime factors of $|G|$.

1: If $|G| = p^a$, then
   - If $p$ is odd, then $p^2 | G$. So $\text{Sup}(G)$ is not LSM.
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2: If $|G| = p^aq^b$, then WLOG $a \geq 2$.
   - If $p$ is odd, then $p^2 | G$, so $\text{Sup}(G)$ is not LSM.
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$\geq 3$: If $|G|$ has at least 3 distinct prime factors, then $pqr | G$, so $\text{Sup}(G)$ is not LSM.
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LSM Proof cont.

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We’d like to thank Dr. Hendrickson for inviting us to research with him and guiding us through the project.

Also, thanks to Concordia College for the financial support.

Lastly, we’d like to thank the organizers of the St. John’s Pi Mu Epsilon conference.
Questions?