Google’s PageRank Algorithm

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Outline

1. The Problem
   - Too many results
   - Principles behind a solution

2. Google Models
   - Five Models
   - Markov Chains

3. Further thoughts
How do you organize an index?

This is two questions, really:

- How do you organize the topics in the index?
- How do you organize the pages found for each topic?

indeterminate, recreational, unsolvable
Ptolemy 3–4, 54, 93
Pythagorean(s) 26, 30, 71n
Pythagorean theorem 19

quartic equation 95, 120, 128–29, 134–35, 140, 140n, 158 (n. 179)
quintic equation 87, 140–41
Qustā ibn Lūqā 37n
al-radd 62
radicals, solution by 140–41
Rahn, J. 124
Recorde, R. 125
recreational problems 25, 64, 93, 97, 102
Regiomontanus, J. 49
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- How do you organize the topics in the index? (Alphabetically)
- How do you organize the pages found for each topic? (Sequentially)
How do you organize a computer search?

For a computer search, you specify the topic, so the first question is irrelevant, but the second still matters.

- Only one topic is desired.
- How do you organize the entries found for a topic?
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The Problem
Google Models
Further thoughts

Drawbacks to returning results sequentially

- Sequential ordering isn’t really informative.
- Okay for rare topics, bad for common topics.
- Slight improvements:
  - Subtopics
  - Markup

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Index of Commands and Con:

- numbering
  - code lines, 172
  - equations
    - resetting the counter, 485
    - subordinate sequences, 484, 485
  - footnotes, 112, 115, 116, 122, 123-125
  - headings, see document headings, numbering
    - lines, 175, 176, 177, 178, 179, 180, 181
    - typed text, 159, 160
  - multilingual documents, 559, 560-563, 564
  - pages, see page numbers
    - sub-numbering float captions, 321, 322, 323
- numberless key (titlesec), 43, 44
- numberless tables of contents, 59
- \numberline, 33, 47, 48, 49, 50-52, 53 (titletoc), 61, 63
- numbers key
  - (fancyverb), 159, 160, 163, 165
  - (listings), 172
- numbers option (natbib), 712, 713, 714, 715
- numbersep key
  - (fancyverb), 159, 160
  - (listings), 172
There *is* no sequential ordering!

Even “rare” topics find many, many webpages. (Example)

Attempted improvements:

- Subtopics—e.g. Yahoo! Directory
- Markup slightly possible—e.g. PDF vs. HTML, showing a few relevant lines
What if we’re searching the Web?

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A solution: ask the librarian

- An expert will know which books are particularly good or relevant.
- Likewise, you could ask people for recommendations of good websites.
- Of course, the quality of the recommendation is only as good as the quality of the recommender.
How does this translate to the web?

**Fundamental Principle**

Every hyperlink on the Web is a recommendation to visit another page.

So one way to figure out which pages are “important” or “good” is to trust the judgment of the other website authors out there who link to it.
How does this translate to the web?

**Fundamental Principle**

Every hyperlink on the Web is a recommendation to visit another page.

- So one way to figure out which pages are “important” or “good” is to trust the judgment of the other website authors out there who link to it.
For simplicity, suppose the web has only six pages, A, B, C, D, E, and F.

There are two natural ways to model its hyperlink structure mathematically.
Modeling the web

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### Directed Graph


### Matrix

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A</strong></td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td><strong>C</strong></td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td><strong>D</strong></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td><strong>E</strong></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td><strong>F</strong></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
### Model 1: Counting hyperlinks

We want to count the number of hyperlinks **into** a webpage.

In other words, we want to count the entries in each row.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>FROM</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>E</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>F</td>
<td>0</td>
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In other words, we want to count the entries in each row.
Matrix Multiplication

As it happens, this is exactly what matrix multiplication by \( \vec{e} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \) does!

\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
\end{pmatrix} =
\begin{pmatrix}
3 \\
1 \\
1 \\
2 \\
0 \\
1 \\
\end{pmatrix}
\]
Problem with Model 1

- Look again at the graph.
- We trust A most, since he has 3 recommendations.
- So when A recommends B, shouldn’t that count for more than E’s recommendation of D?
Model 2: Iteration

Let’s re-run our matrix multiplication, but multiplying each recommendation by the recommender’s previous score.

This is $M(M \vec{e}) = M^2 \vec{e}$. 
Model 2: Iteration

\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
3 \\
1 \\
1 \\
2 \\
0 \\
1 \\
\end{pmatrix}
= 
\begin{pmatrix}
4 \\
3 \\
1 \\
1 \\
0 \\
2 \\
\end{pmatrix}
\]

Let’s re-run our matrix multiplication, but multiplying each recommendation by the recommender’s previous score.

This is \(M(M\vec{e}) = M^2 \vec{e}\).
Let’s re-run our matrix multiplication, but multiplying each recommendation by the recommender’s previous score.

This is $M(M\vec{e}) = M^2\vec{e}$. 

Model 2: Iteration

So let’s iterate! Let $\vec{\pi}_0 = \vec{e}$ and let $\vec{\pi}_{k+1} = M\vec{\pi}_k$.

$$M^1 \vec{e} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$
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$$M^1 \vec{e} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \vec{\pi}_1$$
So let’s iterate! Let $\vec{\pi}_0 = \vec{e}$ and let $\vec{\pi}_{k+1} = M\vec{\pi}_k$.

$$M^2 \vec{e} = \begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix} \begin{pmatrix}
3 \\
1 \\
1 \\
2 \\
0 \\
1
\end{pmatrix}$$
Model 2: Iteration

So let’s iterate! Let $\vec{\pi}_0 = \vec{e}$ and let $\vec{\pi}_{k+1} = M\vec{\pi}_k$.

\[
M^2\vec{e} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} = \vec{\pi}_2
\]
Model 2: Iteration

So let’s iterate! Let $\vec{\pi}_0 = \vec{e}$ and let $\vec{\pi}_{k+1} = M\vec{\pi}_k$.

$$M^3\vec{e} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$$
Model 2: Iteration

So let’s iterate! Let $\vec{\pi}_0 = \vec{e}$ and let $\vec{\pi}_{k+1} = M\vec{\pi}_k$.

$$M^3\vec{e} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 3 \\ 3 \\ 0 \\ 1 \end{pmatrix} = \vec{\pi}_3$$
Model 2: Iteration

So let’s iterate! Let $\vec{\pi}_0 = \vec{e}$ and let $\vec{\pi}_{k+1} = M\vec{\pi}_k$.

$$M^4\vec{e} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \\ 3 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$
So let’s iterate! Let $\vec{\pi}_0 = \vec{e}$ and let $\vec{\pi}_{k+1} = M \vec{\pi}_k$.

$$M^4 \vec{e} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \\ 3 \\ 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ 4 \\ 4 \\ 0 \\ 3 \end{pmatrix} = \vec{\pi}_4$$
Model 2: Iteration

So let’s iterate! Let \( \vec{\pi}_0 = \vec{e} \) and let \( \vec{\pi}_{k+1} = M\vec{\pi}_k \).

\[
M^5 \vec{e} = \begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
10 \\
5 \\
4 \\
4 \\
0 \\
3 \\
\end{pmatrix}
\]
Model 2: Iteration

So let’s iterate! Let $\vec{\pi}_0 = \vec{e}$ and let $\vec{\pi}_{k+1} = M\vec{\pi}_k$.

$$M^5 \vec{e} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 10 \\ 5 \\ 4 \\ 4 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ 10 \\ 5 \\ 5 \\ 0 \\ 4 \end{pmatrix} = \vec{\pi}_5$$
**Model 2: Iteration**

So let’s iterate! Let $\vec{\pi}_0 = \vec{e}$ and let $\vec{\pi}_{k+1} = M\vec{\pi}_k$.

$$M^6\vec{e} = \begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix}
13 \\
10 \\
5 \\
5 \\
0 \\
4 \\
\end{pmatrix}$$
Model 2: Iteration

So let’s iterate! Let $\vec{\pi}_0 = \vec{e}$ and let $\vec{\pi}_{k+1} = M\vec{\pi}_k$.

\[
M^6 \vec{e} = \begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
13 \\
10 \\
5 \\
5 \\
0 \\
4
\end{pmatrix}
= \begin{pmatrix}
20 \\
13 \\
10 \\
10 \\
0 \\
5
\end{pmatrix}
= \vec{\pi}_6
\]
Model 2: Iteration

So let’s iterate! Let $\vec{\pi}_0 = \vec{e}$ and let $\vec{\pi}_{k+1} = M\vec{\pi}_k$.

$$M^7 \vec{e} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 20 \\ 13 \\ 10 \\ 10 \\ 0 \\ 5 \end{pmatrix}$$
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\[
M^7 \vec{e} = \begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
20 \\
13 \\
10 \\
10 \\
0 \\
5
\end{pmatrix}
= \begin{pmatrix}
33 \\
20 \\
13 \\
13 \\
0 \\
10
\end{pmatrix}
= \vec{\pi}_7
\]
So let’s iterate! Let $\vec{\pi}_0 = \vec{e}$ and let $\vec{\pi}_{k+1} = M \vec{\pi}_k$.

\[
M^8 \vec{e} = \begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
33 \\
20 \\
13 \\
13 \\
0 \\
10
\end{pmatrix}
\]
So let’s iterate! Let \( \vec{\pi}_0 = \vec{e} \) and let \( \vec{\pi}_{k+1} = M\vec{\pi}_k \).

\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
33 \\
20 \\
13 \\
13 \\
0 \\
10
\end{pmatrix}
= 
\begin{pmatrix}
46 \\
33 \\
20 \\
20 \\
0 \\
13
\end{pmatrix}
= \vec{\pi}_8
Problems with Model 2

Four problems:

- Where should we stop?
- When we iterate, the numbers get bigger and bigger.
- How do we know the rankings stabilize?
- A profligate recommender will have an undue influence on the results.

Solution: when $B$ recommends 3 people ($A$, $C$, and $D$), let’s make that worth $\frac{1}{3}$ each instead of 1.
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Solution: when $B$ recommends 3 people ($A$, $C$, and $D$), let’s make that worth $\frac{1}{3}$ each instead of 1.
Scaling the columns to sum to 1 gives us a hyperlink matrix $H$:

$$
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}.
$$
Model 3: hyperlink matrix $H$

Scaling the columns to sum to 1 gives us a hyperlink matrix $H$:

$$H = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$
Scaling the columns to sum to 1 gives us a hyperlink matrix $H$:

$$H = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 
\end{pmatrix}.$$

So let's let $\pi_0 = \vec{e}$ and let $\pi_{k+1} = H\pi_k$. 
Model 3: hyperlink matrix $H$

$$\vec{\pi}_0 = H^0 \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}^0 \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.17 \\ 0.17 \\ 0.17 \\ 0.17 \\ 0.17 \\ 0.17 \end{pmatrix}$$
Model 3: hyperlink matrix $H$

$$\vec{\pi}_1 = H^1 \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}^1 \begin{pmatrix} 1 \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.31 \\ 0.17 \\ 0.06 \\ 0.22 \\ 0.00 \\ 0.08 \end{pmatrix}$$
Model 3: hyperlink matrix $H$

$$\vec{\pi}_2 = H^2 \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0
\end{pmatrix}^2 \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6}
\end{pmatrix} = \begin{pmatrix}
0.22 \\
0.31 \\
0.06 \\
0.06 \\
0.00 \\
0.11
\end{pmatrix}$$
## Model 3: hyperlink matrix $H$

The problem:

$$\vec{\pi}_3 = H^3 \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0
\end{pmatrix}^3 \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6}
\end{pmatrix} = \begin{pmatrix}
0.19 \\
0.22 \\
0.10 \\
0.10 \\
0.00 \\
0.03
\end{pmatrix}$$

Why? Because $F$ is a "dangling node."
Model 3: hyperlink matrix $H$

$$\vec{\pi}_4 = H^4 \frac{1}{n} \vec{e} = \left( \begin{array}{ccccccc} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \end{array} \right)^4 \left( \begin{array}{c} 1 \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{array} \right) = \left( \begin{array}{c} 0.23 \\ 0.19 \\ 0.07 \\ 0.07 \\ 0.00 \\ 0.05 \end{array} \right)$$
Model 3: hyperlink matrix $H$

$$\vec{\pi}_5 = H^5 \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}^5 \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.17 \\ 0.23 \\ 0.06 \\ 0.06 \\ 0.00 \\ 0.04 \end{pmatrix}$$
Model 3: hyperlink matrix $H$

$$\vec{\pi}_6 = H^6 \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}^6 \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.17 \\ 0.17 \\ 0.08 \\ 0.08 \\ 0.00 \\ 0.03 \end{pmatrix}$$
Model 3: hyperlink matrix $H$

\[ \vec{\pi}_7 = \frac{1}{n} \vec{e} = H^7 \begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{2} & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.17 \\ 0.17 \\ 0.06 \\ 0.06 \\ 0.00 \\ 0.04 \end{pmatrix} \]
Model 3: hyperlink matrix $H$

$$\vec{\pi}_8 = H^8 \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}^8 \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.14 \\ 0.17 \\ 0.06 \\ 0.06 \\ 0.00 \\ 0.03 \end{pmatrix}$$

Problem: All the scores go to zero!

Why? Because $F$ is "dangling node."
The Problem
Google Models
Further thoughts

Model 3: hyperlink matrix $H$

Model 3: hyperlink matrix $H$

$$\vec{\pi}_9 = H^9 \frac{1}{n} \vec{e} = \left( \begin{array}{cccccc} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{array} \right)^9 \left( \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{array} \right) = \left( \begin{array}{c} 0.14 \\ 0.14 \\ 0.06 \\ 0.06 \\ 0.00 \\ 0.03 \end{array} \right)$$

Anders O. F. Hendrickson
Google’s PageRank Algorithm
The Problem
Google Models
Further thoughts

Five Models
Markov Chains

Model 3: hyperlink matrix $H$

$$\vec{\pi}_{10} = H^{10} \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
\end{pmatrix}^{10} \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\end{pmatrix} = \begin{pmatrix}
0.13 \\
0.14 \\
0.05 \\
0.05 \\
0.00 \\
0.03 \\
\end{pmatrix}$$
Model 3: hyperlink matrix $H$

$\vec{\pi}_{11} = H^{11} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}^{11} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.12 \\ 0.13 \\ 0.05 \\ 0.05 \\ 0.00 \\ 0.02 \end{pmatrix}$
Model 3: hyperlink matrix $H$

$$\vec{\pi}_{12} = H^{12} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}^{12} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.11 \\ 0.12 \\ 0.04 \\ 0.04 \\ 0.00 \\ 0.02 \end{pmatrix}$$
Model 3: hyperlink matrix $H$

$$\vec{\pi}_{13} = H^{13} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}^{13} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.11 \\ 0.11 \\ 0.04 \\ 0.04 \\ 0.00 \\ 0.02 \end{pmatrix}$$
Model 3: hyperlink matrix $H$

$$\vec{\pi}_{14} = H^{14} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}^{14} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.10 \\ 0.11 \\ 0.04 \\ 0.04 \\ 0.00 \\ 0.02 \end{pmatrix}$$
Model 3: hyperlink matrix $H$

$$\vec{\pi}_{15} = H^{15} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}^{15} \begin{pmatrix} 1 \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.09 \\ 0.10 \\ 0.04 \\ 0.04 \\ 0.00 \\ 0.02 \end{pmatrix}$$

Problem: All the scores go to zero!

Why? Because $F$ is "dangling node."
Model 3: hyperlink matrix $H$

$$\vec{\pi}_{16} = \frac{1}{n} \frac{1}{\bar{e}} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}^{16} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.09 \\ 0.09 \\ 0.03 \\ 0.03 \\ 0.00 \\ 0.02 \end{pmatrix}$$

All the scores go to zero!

Why? Because $F$ is "dangling node."
The Problem
Google Models
Further thoughts

Five Models
Markov Chains

Model 3: hyperlink matrix $H$

$\pi_{17} = H^{17} \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
\end{pmatrix}^{17} \begin{pmatrix}
\frac{1}{61} \\
\frac{1}{61} \\
\frac{1}{61} \\
\frac{1}{61} \\
\frac{1}{61} \\
\frac{1}{61} \\
\end{pmatrix} = \begin{pmatrix}
0.08 \\
0.09 \\
0.03 \\
0.03 \\
0.00 \\
0.02 \\
\end{pmatrix}$

Problem: All the scores go to zero!
Why? Because $F$ is “dangling node.”

Anders O. F. Hendrickson
Google’s PageRank Algorithm
Model 3: hyperlink matrix $H$

$$\vec{\pi}_{18} = H^{18} \frac{1}{n} \vec{e} = \left( \begin{array}{cccccc} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{array} \right)^{18} \left( \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{array} \right) = \left( \begin{array}{c} 0.06 \\ 0.08 \\ 0.03 \\ 0.03 \\ 0.00 \\ 0.02 \end{array} \right)$$
Model 3: hyperlink matrix $H$

$$\vec{\pi}_{19} = H^{19} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}^{19} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.06 \\ 0.06 \\ 0.03 \\ 0.03 \\ 0.00 \\ 0.01 \end{pmatrix}$$

Anders O. F. Hendrickson
Google's PageRank Algorithm
Model 3: hyperlink matrix $H$

$$\vec{\pi}_{20} = H^{20} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}^{20} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.05 \\ 0.06 \\ 0.02 \\ 0.02 \\ 0.00 \\ 0.01 \end{pmatrix}$$

Problem: All the scores go to zero!

Problem: All the scores go to zero!
Model 3: hyperlink matrix $H$

\[ \overline{\pi}_{21} = H^{21} \frac{1}{n} \overline{e} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}^{21} \begin{pmatrix} \frac{1}{n} \\ \frac{1}{n} \\ \frac{1}{n} \\ \frac{1}{n} \\ \frac{1}{n} \\ \frac{1}{n} \end{pmatrix} = \begin{pmatrix} 0.05 \\ 0.05 \\ 0.02 \\ 0.02 \\ 0.00 \\ 0.01 \end{pmatrix} \]

- Problem: All the scores go to zero!

- Why? Because $F$ is “dangling node.”
Fix the dangling node

- A dangling node like $F$ recommends no other page.
- Instead, let’s “read” $F$ as recommending everyone equally.

$$H = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
\end{pmatrix} + 1 \cdot \vec{e}$$

where $\vec{a}$ is a vector whose 1's indicate dangling nodes.
Fix the dangling node

- A dangling node like $F$ recommends no other page.
- Instead, let’s “read” $F$ as recommending everyone equally.

\[ S = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6}
\end{pmatrix} \]
A dangling node like $F$ recommends no other page. Instead, let’s “read” $F$ as recommending everyone equally.

\[ S = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
\end{pmatrix} \]
Fix the dangling node

- A dangling node like $F$ recommends no other page.
- Instead, let’s “read” $F$ as recommending everyone equally.

\[
S = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
= H + \frac{1}{6} \begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
A dangling node like $F$ recommends no other page.
Instead, let’s “read” $F$ as recommending everyone equally.

$$S = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \end{pmatrix}$$

$$= H + \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = H + \frac{1}{n} \bar{e} \bar{a}^T.$$

where $\bar{a}$ is a vector whose 1’s indicate dangling nodes.
Model 4: \( S = H + \frac{1}{n} \vec{e} \vec{a}^T \)

\[
\vec{\pi}_0 = S^0 \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} \\
1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6} \\
\end{pmatrix}^0 \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\end{pmatrix} = \begin{pmatrix}
0.17 \\
0.17 \\
0.17 \\
0.17 \\
0.17 \\
0.17 \\
\end{pmatrix}
\]
Model 4: $S = H + \frac{1}{n} \vec{e} \vec{a}^T$

$$\pi_1 = S^1 \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6} \end{pmatrix}^1 \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.33 \\ 0.19 \\ 0.08 \\ 0.25 \\ 0.03 \\ 0.11 \end{pmatrix}$$
Model 4: \( S = H + \frac{1}{n} \vec{e}\vec{a}^T \)

\[
\vec{\pi}_2 = S^2 \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} \\
1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6}
\end{pmatrix}^2 \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6}
\end{pmatrix} = \begin{pmatrix}
0.29 \\
0.35 \\
0.08 \\
0.11 \\
0.02 \\
0.14
\end{pmatrix}
\]
Model 4: \( S = H + \frac{1}{n} \vec{e} \vec{a}^T \)

\[
\vec{\pi}_3 = S^3 \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} \\
1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6} \\
\end{pmatrix}^3 \begin{pmatrix}
\frac{1}{6} \\
\frac{6}{6} \\
\frac{6}{6} \\
\frac{6}{6} \\
\frac{6}{6} \\
\frac{6}{6} \\
\end{pmatrix} = \begin{pmatrix}
0.28 \\
0.32 \\
0.14 \\
0.16 \\
0.02 \\
0.08 \\
\end{pmatrix}
\]
Model 4: $S = H + \frac{1}{n} \vec{e} \vec{a}^T$

$$\vec{\pi}_4 = S^4 \frac{1}{n} \vec{e} = \left( \begin{array}{ccccccc} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} & 1 \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{6} & 1 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} & 1 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 1 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6} & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 1 \end{array} \right)^4 \left( \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{array} \right) = \left( \begin{array}{c} 0.34 \\ 0.29 \\ 0.12 \\ 0.14 \\ 0.01 \\ 0.09 \end{array} \right)$$
Model 4: \( S = H + \frac{1}{n} \vec{e} \vec{a}^T \)

\[
\vec{\pi}_5 = S^5 \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} \\
1 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6}
\end{pmatrix}^5 \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6}
\end{pmatrix} = \begin{pmatrix}
0.30 \\
0.36 \\
0.11 \\
0.13 \\
0.02 \\
0.09
\end{pmatrix}
\]
Model 4: \( S = H + \frac{1}{n} \vec{e} \vec{a}^T \)

\[
\vec{\pi}_6 = S^6 \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} \\
1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6} \\
\end{pmatrix}^6 \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\end{pmatrix} = \begin{pmatrix}
0.31 \\
0.32 \\
0.13 \\
0.15 \\
0.01 \\
0.08 \\
\end{pmatrix}
\]
Model 4: \[ S = H + \frac{1}{n} \vec{e} \vec{a}^T \]

\[
\bar{\pi}_7 = S^7 \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6} \end{pmatrix}^7 \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.33 \\ 0.32 \\ 0.12 \\ 0.13 \\ 0.01 \\ 0.09 \end{pmatrix}

Anders O. F. Hendrickson
Google's PageRank Algorithm
Model 4: \( S = H + \frac{1}{n}\vec{e}\vec{a}^T \)

\[
\vec{\pi}_8 = S^8 \frac{1}{n}\vec{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} \\
1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6} \\
\end{pmatrix}^8 \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\end{pmatrix} = \begin{pmatrix}
0.31 \\
0.34 \\
0.12 \\
0.13 \\
0.01 \\
0.08 \\
\end{pmatrix}
\]
Model 4: \( S = H + \frac{1}{n} \vec{e} \vec{a}^T \)

\[
\vec{\pi}_9 = S^9 \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} \\
1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6}
\end{pmatrix}^9 \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6}
\end{pmatrix} = \begin{pmatrix}
0.32 \\
0.32 \\
0.13 \\
0.14 \\
0.01 \\
0.08
\end{pmatrix}
\]
Model 4: $S = H + \frac{1}{n} \vec{e} \vec{a}^T$

$\pi_{10} = S^{10} \frac{1}{n} \vec{e} = \left( \begin{array}{cccccc} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6} \end{array} \right) ^{10} \left( \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{array} \right) = \left( \begin{array}{c} 0.32 \\ 0.33 \\ 0.12 \\ 0.13 \\ 0.01 \\ 0.08 \end{array} \right)$
Model 4: \( S = H + \frac{1}{n} \vec{e} \vec{a}^T \)

\[
\tilde{\pi}_{11} = S^{11} \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} & 1 \\
1 & 0 & 0 & 0 & 0 & \frac{6}{6} & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{6}{6} & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 1 & \frac{6}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{6}{6} & \frac{1}{6} \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{6}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{6}{6} & \frac{1}{6}
\end{pmatrix}^{11} \begin{pmatrix}
1 \\
\frac{1}{6} \\
\frac{6}{6} \\
\frac{6}{6} \\
\frac{6}{6} \\
\frac{6}{6} \\
\frac{6}{6}
\end{pmatrix} = \begin{pmatrix}
0.31 \\
0.33 \\
0.12 \\
0.14 \\
0.01 \\
0.08
\end{pmatrix}
\]
Model 4: \( S = H + \frac{1}{n} \vec{e} \vec{a}^T \)

\[
\vec{\pi}_{12} = S^{12} \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} \\
1 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6}
\end{pmatrix}^{12} \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6}
\end{pmatrix} = \begin{pmatrix}
0.32 \\
0.32 \\
0.12 \\
0.14 \\
0.01 \\
0.08
\end{pmatrix}
\]
Model 4: $S = H + \frac{1}{n} \vec{e} \vec{a}^T$

$$\tilde{\pi}_{13} = S^{13} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} & 0 \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6} \end{pmatrix}^{13} \begin{pmatrix} 1 \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.32 \\ 0.33 \\ 0.12 \\ 0.14 \\ 0.01 \\ 0.08 \end{pmatrix}$$

Anders O. F. Hendrickson  
Google's PageRank Algorithm
Model 4: \[ S = H + \frac{1}{n} \vec{e} \vec{a}^T \]

\[ \vec{\pi}_{14} = S^{14} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & 1/3 & 1 & 1/2 & 0 & 1/6 & 1/6 & 1/6 & 1/6 \ 1 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 \ 0 & 1/3 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 \ 0 & 1/3 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 \ 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 \ 0 & 0 & 0 & 1/2 & 0 & 1/6 & 1/6 & 1/6 & 1/6 \ \end{pmatrix}^{14} \begin{pmatrix} 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \end{pmatrix} = \begin{pmatrix} 0.31 \\ 0.33 \\ 0.12 \\ 0.14 \\ 0.01 \\ 0.08 \end{pmatrix} \]
Model 4: \[ S = H + \frac{1}{n} \vec{e} \vec{a}^T \]

\[ \vec{\pi}_{15} = S^{15} \frac{1}{n} \vec{e} = \left( \begin{array}{cccc} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6} \end{array} \right)^{15} \left( \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{array} \right) = \left( \begin{array}{c} 0.32 \\ 0.33 \\ 0.12 \\ 0.14 \\ 0.01 \\ 0.08 \end{array} \right) \]
Model 4: \( S = H + \frac{1}{n} \vec{e} \vec{a}^T \)

\[
\vec{\pi}_{16} = S^{16} \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} \\
1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6}
\end{pmatrix} \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6}
\end{pmatrix} = \begin{pmatrix}
0.31 \\
0.33 \\
0.12 \\
0.14 \\
0.01 \\
0.08
\end{pmatrix}
\]
Model 4: \( S = H + \frac{1}{n} \vec{e} \vec{a}^T \)

\[
\vec{\pi}_{17} = S^{17} \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
1 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 1 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\end{pmatrix}^{17} \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\end{pmatrix} = \begin{pmatrix}
0.31 \\
0.33 \\
0.12 \\
0.14 \\
0.01 \\
0.08 \\
\end{pmatrix}
\]
Model 4: $S = H + \frac{1}{n} \vec{e} \vec{a}^T$

$$\vec{\pi}_{18} = S^{18} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6} \end{pmatrix}^{18} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.32 \\ 0.33 \\ 0.12 \\ 0.14 \\ 0.01 \\ 0.08 \end{pmatrix}$$
Model 4: \( S = H + \frac{1}{n} \vec{e} \vec{a}^T \)

\[
\vec{\pi}_{19} = S^{19} \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} & 1 \\
1 & 0 & 0 & 0 & 0 & \frac{1}{6} & 1 \\
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} & 1 \\
0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} & 1 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 1 \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6} & 1
\end{pmatrix}^{19} \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6}
\end{pmatrix} = \begin{pmatrix}
0.31 \\
0.33 \\
0.12 \\
0.14 \\
0.01 \\
0.08
\end{pmatrix}
\]
Model 4: \[ S = H + \frac{1}{n} \bar{e} \bar{a}^T \]

\[ \bar{\pi}_{20} = S^{20} \frac{1}{n} \bar{e} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6} \end{pmatrix}^{20} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.32 \\ 0.33 \\ 0.12 \\ 0.14 \\ 0.01 \\ 0.08 \end{pmatrix} \]
Model 4: \( S = H + \frac{1}{n} \vec{e} \vec{a}^T \)

\[ \vec{\pi}_{21} = S^{21} \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} \\
1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6}
\end{pmatrix}^{21} \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6}
\end{pmatrix} = \begin{pmatrix}
0.32 \\
0.33 \\
0.12 \\
0.14 \\
0.01 \\
0.08
\end{pmatrix} \]
Model 4: \[ S = H + \frac{1}{n} \bar{e}a^T \]

\[ \bar{\pi}_{22} = S^{22} \frac{1}{n} \bar{e} = \left[ \begin{array}{cccccc} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6} \end{array} \right]^{22} \left[ \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{array} \right] = \left[ \begin{array}{c} 0.31 \\ 0.33 \\ 0.12 \\ 0.14 \\ 0.01 \\ 0.08 \end{array} \right] 

Hooray!
Model 4: \[ S = H + \frac{1}{n} \vec{e} \vec{a}^T \]

\[ \vec{\pi}_{23} = S^{23} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6} \end{pmatrix}^{23} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.32 \\ 0.33 \\ 0.12 \\ 0.14 \\ 0.01 \\ 0.08 \end{pmatrix} \]

Hooray!
Model 4: $S = H + \frac{1}{n} \vec{e} \vec{a}^T$

$$\vec{\pi}_{24} = S^{24} \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} \\
1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6}
\end{pmatrix}^{24} \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6}
\end{pmatrix} = \begin{pmatrix}
0.32 \\
0.33 \\
0.12 \\
0.14 \\
0.01 \\
0.08
\end{pmatrix}$$

Hooray!
Model 4: \( S = H + \frac{1}{n} \vec{e} \vec{a}^T \)

\[
\vec{\pi}_{25} = S^{25} \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & \frac{1}{2} & 0
\end{pmatrix}^{25} \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6}
\end{pmatrix} = \begin{pmatrix}
0.32 \\
0.33 \\
0.12 \\
0.14 \\
0.01 \\
0.08
\end{pmatrix}
\]

Hooray!
Model 4: $S = H + \frac{1}{n} \bar{e} \bar{a}^T$

$\pi_{26} = S^{26} \frac{1}{n} \bar{e} = \begin{pmatrix}
0 & \frac{1}{3} & 1 & \frac{1}{2} & 0 & \frac{1}{6} \\
1 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{1}{3} & 0 & 0 & 1 & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{6}
\end{pmatrix}^{26} \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6}
\end{pmatrix} = \begin{pmatrix}
0.32 \\
0.33 \\
0.12 \\
0.14 \\
0.01 \\
0.08
\end{pmatrix}$

Hooray!
Problem with Model 4: cycles

\[ \vec{\pi}_0 = S^0 \frac{1}{n} \vec{e} = \left( \begin{array}{cccccc} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right) \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.17 \\ 0.17 \\ 0.17 \\ 0.17 \\ 0.17 \end{pmatrix} \]
The Problem
Google Models
Further thoughts

Problem with Model 4: cycles

\[ \vec{\pi}_1 = S^1 \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^1 \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.22 \\ 0.17 \\ 0.17 \\ 0.06 \\ 0.22 \\ 0.17 \end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_2 = S^2 \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \end{pmatrix}^2 \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.11 \\ 0.22 \\ 0.17 \\ 0.06 \\ 0.22 \\ 0.22 \end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_3 = S^3 \frac{1}{n} \vec{e} = \left( \begin{array}{cccccc} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right)^3 \left( \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{array} \right) = \left( \begin{array}{c} 0.11 \\ 0.11 \\ 0.22 \\ 0.06 \\ 0.28 \\ 0.22 \end{array} \right) \]
Problem with Model 4: cycles

\[ \vec{\pi}_4 = S^4 \frac{1}{n} \vec{\epsilon} = \left( \begin{array}{cccccc} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right)^4 \left( \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{array} \right) = \left( \begin{array}{c} 0.13 \\ 0.11 \\ 0.11 \\ 0.07 \\ 0.30 \\ 0.28 \end{array} \right) \]
Problem with Model 4: cycles

\[\vec{\pi}_5 = S^5 \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^5 \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \end{pmatrix} = \begin{pmatrix}
0.11 \\
0.13 \\
0.11 \\
0.04 \\
0.31 \\
0.30 \end{pmatrix}\]
Problem with Model 4: cycles

\[ \vec{\pi}_6 = S^6 \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^6 \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.07 \\ 0.11 \\ 0.13 \\ 0.04 \\ 0.33 \\ 0.31 \end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_7 = S^7 \frac{1}{n} \vec{e} = \left( \begin{array}{cccccc} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right)^7 \left( \begin{array}{c} 1 \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{array} \right) = \left( \begin{array}{c} 0.08 \\ 0.07 \\ 0.11 \\ 0.04 \\ 0.36 \\ 0.33 \end{array} \right) \]
Problem with Model 4: cycles

\[ \vec{\pi}_8 = S^8 \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^8 \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.08 \\ 0.08 \\ 0.07 \\ 0.04 \\ 0.37 \\ 0.36 \end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_9 = S^9 \frac{1}{n} \vec{e} = \left( \begin{array}{cccccc} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right)^9 \left( \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{array} \right) = \left( \begin{array}{c} 0.06 \\ 0.08 \\ 0.08 \\ 0.02 \\ 0.38 \\ 0.37 \end{array} \right) \]
Problem with Model 4: cycles

\[ \vec{\pi}_{10} = S^{10} \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}^{10} \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6}
\end{pmatrix} = \begin{pmatrix}
0.05 \\
0.06 \\
0.08 \\
0.03 \\
0.40 \\
0.38
\end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_{11} = S^{11} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^{11} \begin{pmatrix} 1 \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.05 \\ 0.05 \\ 0.06 \\ 0.03 \\ 0.41 \\ 0.40 \end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_{12} = S^{12} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^{12} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.05 \\ 0.05 \\ 0.05 \\ 0.02 \\ 0.42 \\ 0.41 \end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_{13} = S^{13} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^{13} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.04 \\ 0.05 \\ 0.05 \\ 0.02 \\ 0.43 \\ 0.42 \end{pmatrix} \]
Problem with Model 4: cycles

\[ \pi_{14} = S^{14} \frac{1}{n} \mathbf{e} = \left( \begin{array}{cccccc} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right)^{14} \left( \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{array} \right) = \left( \begin{array}{c} 0.03 \\ 0.04 \\ 0.05 \\ 0.02 \\ 0.44 \\ 0.43 \end{array} \right) \]
Problem with Model 4: cycles

\[ \vec{\pi}_{15} = S^{15} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^{15} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.03 \\ 0.03 \\ 0.04 \\ 0.02 \\ 0.44 \\ 0.44 \end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_{16} = S^{16} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^{16} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.03 \\ 0.03 \\ 0.03 \\ 0.01 \\ 0.45 \\ 0.44 \end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_{17} = S^{17} \frac{1}{n} \vec{e} = \left( \begin{array}{cccccc} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right)^{17} \left( \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{array} \right) = \left( \begin{array}{c} 0.02 \\ 0.03 \\ 0.03 \\ 0.01 \\ 0.45 \\ 0.45 \end{array} \right) \]
Problem with Model 4: cycles

\[ \vec{\pi}_{18} = S^{18} \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{18} \begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \end{pmatrix} = \begin{pmatrix}
0.02 \\
0.02 \\
0.03 \\
0.01 \\
0.46 \\
0.45 \end{pmatrix} \]
Problem with Model 4: cycles

$$\tilde{\pi}_{19} = S^{19} \frac{1}{n} \tilde{e} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^{19} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.02 \\ 0.02 \\ 0.02 \\ 0.01 \\ 0.46 \\ 0.46 \end{pmatrix}$$
Problem with Model 4: cycles

\[ \vec{\pi}_{20} = S^{20} \frac{1}{n} \vec{e} = \left( \begin{array}{cccccc} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right)^{20} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.02 \\ 0.02 \\ 0.02 \\ 0.01 \\ 0.47 \\ 0.46 \end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_{21} = S^{21} \frac{1}{n} \vec{\epsilon} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^{21} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.02 \\ 0.02 \\ 0.02 \\ 0.01 \\ 0.47 \\ 0.47 \end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_{22} = S^{22} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^{22} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.01 \\ 0.02 \\ 0.02 \\ 0.01 \\ 0.47 \\ 0.47 \end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_{23} = S^{23} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^{23} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.01 \\ 0.01 \\ 0.02 \\ 0.01 \\ 0.48 \\ 0.47 \end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_{24} = S^{24} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^{24} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.48 \\ 0.48 \end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_{25} = S^{25} \frac{1}{n} \vec{e} = \left( \begin{array}{cccccc} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right)^{25} \left( \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{array} \right) = \left( \begin{array}{c} 0.01 \\ 0.01 \\ 0.01 \\ 0.00 \\ 0.48 \\ 0.48 \end{array} \right) \]
Problem with Model 4: cycles

\[ \vec{\pi}_{26} = S^{26} \frac{1}{n} \vec{e} = \left( \begin{array}{cccccc} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right)^{26} \left( \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{array} \right) = \left( \begin{array}{c} 0.01 \\ 0.01 \\ 0.01 \\ 0.00 \\ 0.48 \\ 0.48 \end{array} \right) \]
Problem with Model 4: cycles

\[
\vec{\pi}_{27} = S^{27} \frac{1}{n} \vec{e} =
\begin{pmatrix}
0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6}
\end{pmatrix}
= 
\begin{pmatrix}
0.01 \\
0.01 \\
0.01 \\
0.00 \\
0.49 \\
0.48
\end{pmatrix}
\]
Problem with Model 4: cycles

\[ \vec{\pi}_{28} = S^{28} \frac{1}{n} \vec{e} = \left( \begin{array}{cccccc} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right)^{28} \left( \begin{array}{c} 1 \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{array} \right) = \left( \begin{array}{c} 0.01 \\ 0.01 \\ 0.01 \\ 0.00 \\ 0.49 \\ 0.49 \end{array} \right) \]
Problem with Model 4: cycles

\[ \vec{\pi}_{29} = S^{29} \frac{1}{n} \vec{e} = \begin{pmatrix}
0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}^{29} \begin{pmatrix} 1 \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.01 \\ 0.01 \\ 0.01 \\ 0.00 \\ 0.49 \\ 0.49 \end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_{30} = S^{30} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^{30} \begin{pmatrix} 1 \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.01 \\ 0.01 \\ 0.01 \\ 0.00 \\ 0.49 \\ 0.49 \end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_{31} = S^{31} \frac{1}{n} \vec{e} = \left( \begin{array}{cccccc} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right)^{31} \left( \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{array} \right) = \left( \begin{array}{c} 0.00 \\ 0.01 \\ 0.01 \\ 0.00 \\ 0.49 \\ 0.49 \end{array} \right) \]
Problem with Model 4: cycles

\[ \vec{\pi}_{32} = S^{32} \frac{1}{n} \vec{e} = \left( \begin{array}{cccc} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right)^{32} \left( \begin{array}{c} 1 \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{array} \right) = \left( \begin{array}{c} 0.00 \\ 0.00 \\ 0.01 \\ 0.00 \\ 0.49 \\ 0.49 \end{array} \right) \]
Problem with Model 4: cycles

\[ \pi_{33}^{33} = S^{33} \frac{1}{n} \tilde{e} = \begin{pmatrix}
0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}^{33} \begin{pmatrix}
1 \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6}
\end{pmatrix} = \begin{pmatrix}
0.00 \\
0.00 \\
0.00 \\
0.49 \\
0.49
\end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_{34} = S^{34} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{34} \begin{pmatrix} 1 \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.00 \\ 0.00 \\ 0.00 \\ 0.49 \\ 0.49 \end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_{35} = S^{35} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{35} \begin{pmatrix} 1 \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.00 \\ 0.00 \\ 0.00 \\ 0.49 \\ 0.49 \end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_{36} = S^{36} \frac{1}{n} \vec{e} = \left( \begin{array}{cccccc}
0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{array} \right)^{36} \left( \begin{array}{c}
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\frac{1}{6} \\
\end{array} \right) = \left( \begin{array}{c}
0.00 \\
0.00 \\
0.00 \\
0.00 \\
0.49 \\
0.49 \\
\end{array} \right) \]
Problem with Model 4: cycles

\[ \vec{\pi}_{37} = S^{37} \frac{1}{n} \vec{e} = \left( \begin{array}{ccccccc} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right)^{37} \left( \begin{array}{c} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{array} \right) = \left( \begin{array}{c} 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \\ 0.50 \\ 0.49 \end{array} \right) \]

Bother.
Problem with Model 4: cycles

\[ \vec{\pi}_{38} = S^{38} \frac{1}{n} \vec{e} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^{38} \begin{pmatrix} 1 \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 0.00 \\ 0.00 \\ 0.00 \\ 0.50 \\ 0.50 \end{pmatrix} \]
Problem with Model 4: cycles

\[ \vec{\pi}_{39} = S^{39} \frac{1}{n} \vec{e} = \begin{pmatrix}
    0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\
    1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\
    0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\
    0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}^{39} \begin{pmatrix}
    1 \\
    \frac{1}{6} \\
    \frac{1}{6} \\
    \frac{1}{6} \\
    \frac{1}{6} \\
    \frac{1}{6}
\end{pmatrix} = \begin{pmatrix}
    0.00 \\
    0.00 \\
    0.00 \\
    0.00 \\
    0.50 \\
    0.50
\end{pmatrix} \]

Bother.
In this graph, $E$ and $F$ form a cycle. Neither is dangling, but together they absorb all the prestige.

The fix: give some probability of “teleporting” from all nodes, not just dangling ones.
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The fix: give some probability of “teleporting” from all nodes, not just dangling ones.
At each turn, every node will share only $\alpha \approx 85\%$ of its prestige equally among the pages it links to, as it did before.

The remaining $1 - \alpha \approx 15\%$ of its prestige is shared equally among all nodes, just like with dangling nodes.

This gives us a new matrix, the “Google matrix”

$$G = \alpha S + (1 - \alpha) \left( \frac{1}{n} \bar{e} \bar{e}^T \right).$$
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The Solution: The Google Matrix

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Model 5: \( G = \alpha S + (1 - \alpha) \frac{1}{n} \bar{e} \bar{e}^T \), \( \alpha = 0.85 \)

\[ \bar{\pi}_0 = G^0 \frac{1}{n} \bar{e} = \begin{pmatrix} 0.17 \\ 0.17 \\ 0.17 \\ 0.17 \\ 0.17 \end{pmatrix} \]

Splendid! It converged fast, too!
Model 5: $G = \alpha S + (1 - \alpha) \frac{1}{n} \vec{e}\vec{e}^T$, $\alpha = 0.85$

$$\vec{\pi}_1 = G^1 \frac{1}{n} \vec{e} = \begin{pmatrix} 0.21 \\ 0.17 \\ 0.17 \\ 0.07 \\ 0.21 \\ 0.17 \end{pmatrix}$$

Splendid! It converged fast, too!
Model 5: \[ G = \alpha S + (1 - \alpha) \frac{1}{n} \bar{e}\bar{e}^T, \quad \alpha = 0.85 \]

\[ \bar{\pi}_2 = G^2 \frac{1}{n} \bar{e} = \begin{pmatrix} 0.13 \\ 0.21 \\ 0.17 \\ 0.07 \\ 0.21 \\ 0.21 \end{pmatrix} \]
Model 5: $G = \alpha S + (1 - \alpha) \frac{1}{n} \vec{e} \vec{e}^T$, $\alpha = 0.85$

$$\vec{\pi}_3 = G^3 \frac{1}{n} \vec{e} = \begin{pmatrix} 0.13 \\ 0.14 \\ 0.20 \\ 0.07 \\ 0.25 \\ 0.21 \end{pmatrix}$$

Splendid! It converged fast, too!
Model 5: $G = \alpha S + (1 - \alpha)\frac{1}{n}\bar{e}\bar{e}^T$, $\alpha = 0.85$

$$\bar{\pi}_4 = G^4 \frac{1}{n}\bar{e} = \begin{pmatrix} 0.14 \\ 0.14 \\ 0.14 \\ 0.08 \\ 0.26 \\ 0.24 \end{pmatrix}$$

Splendid! It converged fast, too!
Model 5: \( G = \alpha S + (1 - \alpha) \frac{1}{n} \bar{e} \bar{e}^T \), \( \alpha = 0.85 \)

\[ \bar{\pi}_5 = G^5 \frac{1}{n} \bar{e} = \begin{pmatrix} 0.14 \\ 0.15 \\ 0.14 \\ 0.07 \\ 0.27 \\ 0.24 \end{pmatrix} \]

Splendid! It converged fast, too!
Model 5: \[ G = \alpha S + (1 - \alpha) \frac{1}{n} \bar{e} \bar{e}^T, \quad \alpha = 0.85 \]

\[ \vec{\pi}_6 = G^6 \frac{1}{n} \vec{e} = \begin{pmatrix} 0.12 \\ 0.14 \\ 0.15 \\ 0.07 \\ 0.27 \\ 0.25 \end{pmatrix} \]

Splendid! It converged fast, too!

Anders O. F. Hendrickson

Google's PageRank Algorithm
Model 5: $G = \alpha S + (1 - \alpha) \frac{1}{n} \bar{e} \bar{e}^T$, $\alpha = 0.85$

$$\vec{\pi}_7 = G^7 \frac{1}{n} \bar{e} = \begin{pmatrix} 0.12 \\ 0.13 \\ 0.14 \\ 0.07 \\ 0.28 \\ 0.26 \end{pmatrix}$$

Splendid! It converged fast, too!
Model 5: $G = \alpha S + (1 - \alpha) \frac{1}{n} \bar{e} \bar{e}^T$, $\alpha = 0.85$

\[ \bar{\pi}_8 = G^8 \frac{1}{n} \bar{e} = \begin{pmatrix} 0.12 \\ 0.13 \\ 0.13 \\ 0.07 \\ 0.28 \\ 0.26 \end{pmatrix} \]

Splendid! It converged fast, too!
Model 5: $G = \alpha S + (1 - \alpha) \frac{1}{n} \vec{e} \vec{e}^T$, $\alpha = 0.85$

$$\vec{\pi}_9 = G^9 \frac{1}{n} \vec{e} = \begin{pmatrix} 0.12 \\ 0.13 \\ 0.14 \\ 0.06 \\ 0.29 \\ 0.27 \end{pmatrix}$$

Splendid! It converged fast, too!
Model 5: $G = \alpha S + (1 - \alpha)\frac{1}{n}\vec{e}\vec{e}^T$, $\alpha = 0.85$

\[ \vec{\pi}_{10} = G^{10}\frac{1}{n}\vec{e} = \begin{pmatrix} 0.12 \\ 0.13 \\ 0.14 \\ 0.06 \\ 0.29 \\ 0.27 \end{pmatrix} \]

Splendid! It converged fast, too!
Model 5: \( G = \alpha S + (1 - \alpha) \frac{1}{n} \bar{e} \bar{e}^T, \alpha = 0.85 \)

\[
\bar{\pi}_{11} = G^{11} \frac{1}{n} \bar{e} = \begin{pmatrix}
0.12 \\
0.12 \\
0.13 \\
0.06 \\
0.29 \\
0.27
\end{pmatrix}
\]

Splendid! It converged fast, too!
Model 5: \( G = \alpha S + (1 - \alpha) \frac{1}{n} \vec{e} \vec{e}^T, \alpha = 0.85 \)

\[ \vec{\pi}_{12} = G^{12} \frac{1}{n} \vec{e} = \begin{pmatrix} 0.12 \\ 0.12 \\ 0.13 \\ 0.06 \\ 0.29 \\ 0.27 \end{pmatrix} \]

Splendid! It converged fast, too!
Model 5: \( G = \alpha S + (1 - \alpha) \frac{1}{n} \vec{e}\vec{e}^T, \alpha = 0.85 \)

\[ \bar{\pi}_{13} = G^{13} \frac{1}{n} \vec{e} = \begin{pmatrix} 0.12 \\ 0.12 \\ 0.13 \\ 0.06 \\ 0.29 \\ 0.27 \end{pmatrix} \]

Splendid! It converged fast, too!
Model 5: $G = \alpha S + (1 - \alpha) \frac{1}{n} \vec{e}\vec{e}^T$, $\alpha = 0.85$

$$\vec{\pi}_{14} = G^{14} \frac{1}{n} \vec{e} = \begin{pmatrix} 0.11 \\ 0.12 \\ 0.13 \\ 0.06 \\ 0.30 \\ 0.28 \end{pmatrix}$$

Splendid! It converged fast, too!
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$\vec{\pi}_{15} = G^{15} \frac{1}{n} \vec{e} = \begin{pmatrix} 0.11 \\ 0.12 \\ 0.13 \\ 0.06 \\ 0.30 \\ 0.28 \end{pmatrix}$

Splendid! It converged fast, too!
Model 5: \( G = \alpha S + (1 - \alpha) \frac{1}{n} \bar{e} \bar{e}^T, \alpha = 0.85 \)

\[
\pi_{16} = G^{16} \frac{1}{n} \bar{e} = \begin{pmatrix}
0.11 \\
0.12 \\
0.13 \\
0.06 \\
0.30 \\
0.28
\end{pmatrix}
\]
Model 5: \( G = \alpha S + (1 - \alpha) \frac{1}{n} \vec{e} \vec{e}^T \), \( \alpha = 0.85 \)

\[ \vec{\pi}_{17} = G^{17} \frac{1}{n} \vec{e} = \begin{pmatrix} 0.11 \\ 0.12 \\ 0.13 \\ 0.06 \\ 0.30 \\ 0.28 \end{pmatrix} \]

Splendid! It converged fast, too!
Model 5: \( G = \alpha S + (1 - \alpha)\frac{1}{n}\vec{e}\vec{e}^T, \ \alpha = 0.85 \)

\[
\vec{\pi}_{18} = G^{18}\frac{1}{n}\vec{e} = \begin{pmatrix}
0.11 \\
0.12 \\
0.13 \\
0.06 \\
0.30 \\
0.28
\end{pmatrix}
\]

Splendid! It converged fast, too!
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$$\vec{\pi}_{19} = G^{19} \frac{1}{n} \vec{e} = \begin{pmatrix} 0.11 \\ 0.12 \\ 0.13 \\ 0.06 \\ 0.30 \\ 0.28 \end{pmatrix}$$

Splendid! It converged fast, too!
This situation is called a Markov chain. Notice that
- $G$ is square, every entry of $G$ is nonnegative, and each column of $G$ sums to 1. So $G$ is stochastic.
- Every entry of $G$ is greater than zero. This implies that $G$ is regular.

**Regular Markov Chain Theorem**

If $G$ is a regular stochastic matrix, then there exists a limiting vector $\vec{\pi}$ such that

$$\lim_{k\to\infty} G^k \vec{x} = \vec{\pi}$$

for all starting vectors $\vec{x}$. 
Markov Chains

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The Problem  
Google Models  
Further thoughts  

Markov Chains

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for all starting vectors \( \mathbf{x} \).
Another way to phrase this is in terms of eigenvectors.

**Definition**

Let $G$ be a square matrix. If $\vec{x}$ is a vector and $\lambda$ a scalar such that $G\vec{x} = \lambda \vec{x}$, we say $\lambda$ is an *eigenvalue* and $\vec{x}$ an *eigenvector* for $G$.

**Facts**

In a regular stochastic matrix like $G$,

- $G\vec{\pi} = \vec{\pi}$

So the limiting vector $\vec{\pi}$ is an eigenvector with eigenvalue $1$.

- In fact, $1$ is the *largest* eigenvalue of $G$. 

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Google's PageRank Algorithm
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Google’s PageRank Algorithm
The matrix $G$ is dense—all positive entries.

Naively, multiplying $G\vec{\pi}$ requires $O(n^2)$ operations.

However, we never actually need $G$ to compute:

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$$\vec{\pi}_{k+1} = G\vec{\pi}_k$$

$$= \left( \alpha \mathbf{S} + (1 - \alpha) \frac{1}{n} \vec{e}\vec{e}^T \right) \vec{\pi}_k$$
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$$= \left( \alpha S + (1 - \alpha) \frac{1}{n} \vec{e}\vec{e}^T \right) \vec{\pi}_k$$

$$= \left( \alpha H + \frac{1}{n} \vec{e}\vec{a}^T + \frac{1 - \alpha}{n} \vec{e}\vec{e}^T \right) \vec{\pi}_k$$

Since $H$ is an extremely sparse matrix, this can be done in $O(n)$ time!
Computation

- The matrix $G$ is *dense*—all positive entries.
- Naively, multiplying $G\vec{\pi}$ requires $O(n^2)$ operations.
- However, we never actually need $G$ to compute:

\[
\vec{\pi}_{k+1} = G\vec{\pi}_k
\]
\[
= \left( \alpha \mathbf{S} + (1 - \alpha) \frac{1}{n} \vec{e} \vec{e}^T \right) \vec{\pi}_k
\]
\[
= \left( \alpha \mathbf{H} + \frac{1}{n} \vec{e} \vec{a}^T + \frac{1 - \alpha}{n} \vec{e} \vec{e}^T \right) \vec{\pi}_k
\]
\[
= \alpha \mathbf{H} \vec{\pi}_k + \frac{1}{n} \vec{e} \vec{a}^T \vec{\pi}_k + \frac{1 - \alpha}{n} \vec{e} \vec{e}^T \vec{\pi}_k
\]
Computation

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- However, we never actually need $G$ to compute:

$$
\vec{\pi}_{k+1} = G\vec{\pi}_k
$$

$$
= \left( \alpha S + (1 - \alpha)\frac{1}{n}\vec{e}\vec{e}^T \right)\vec{\pi}_k
$$

$$
= \left( \alpha H + \frac{1}{n}\vec{e}\vec{a}^T + \frac{1 - \alpha}{n}\vec{e}\vec{e}^T \right)\vec{\pi}_k
$$

$$
= \alpha H\vec{\pi}_k + \frac{1}{n}\vec{e}\vec{a}^T\vec{\pi}_k + \frac{1 - \alpha}{n}\vec{e}\vec{e}^T\vec{\pi}_k
$$

$$
= \alpha H\vec{\pi}_k + \frac{1}{n}(\vec{a} \cdot \vec{\pi}_k + 1 - \alpha)\vec{e}
$$

Since $H$ is an extremely sparse matrix, this can be done in $O(n)$ time!
Computation

- The matrix $G$ is *dense*—all positive entries.
- Naively, multiplying $G\vec{\pi}$ requires $O(n^2)$ operations.
- However, we never actually need $G$ to compute:

$$
\vec{\pi}_{k+1} = G\vec{\pi}_k
$$

$$
= \left( \alpha S + (1 - \alpha) \frac{1}{n} \vec{e}\vec{e}^T \right) \vec{\pi}_k
$$

$$
= \left( \alpha H + \frac{1}{n} \vec{e}\vec{a}^T + \frac{1 - \alpha}{n} \vec{e}\vec{e}^T \right) \vec{\pi}_k
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- Since $H$ is an extremely sparse matrix, this can be done in $O(n)$ time!
The $\alpha$ factor

\[ G = \alpha S + (1 - \alpha) \frac{1}{n} \bar{e} \bar{e}^T. \]

The $\alpha$ factor needs to be carefully chosen!

- The higher $\alpha$ is, the more you’re using the hyperlink structure of the Web.
- So if $\alpha \approx 0$, your ranking will be nearly useless.
- But if $\alpha \approx 1$, your answer will
  - Converge more slowly, and
  - Fluctuate rapidly as links are created or destroyed.
- Google has found $\alpha \approx 0.85$ to work well.
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Everything I’ve said been made public by Google’s founders, Larry Page and Sergey Brin. But much of Google’s power comes from other factors:

- “Relevance” score
- Geographic targeting
- Personalization
- Plenty of secrets
Problems with Google

- Is popularity really the best way to rank websites?
  - Example: Netflix search engine
  - Example: Jim Jefson
  - Example: Anders Hendrickson
- Google bombs
  - Example: French military victories
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Questions?
References

- Java applets by Tim Chartier