Study Guide for Final Exam  
Calculus II, Spring 2009

Questions on the final exam will be taken from the following list of topics.

Chapter 6: Applications of the Integral

- **Basic Principle:** For each and every problem in Chapter 6, you set up the integral using the same three steps:
  1. Slice up the thing into thin slices or rings or shells
  2. Find the value (i.e. area or volume or work) of each thin piece
  3. Add up all those values by integrating.

As long as you follow these three steps and use your common sense, you should be able to set up the integral for every problem.

- **The area between two curves** is thus given by

  \[
  \int_a^b (y_{\text{top}} - y_{\text{bottom}}) \, dx \quad \text{(vertical slices)} \quad \text{or} \quad \int_a^b (x_{\text{right}} - x_{\text{left}}) \, dy \quad \text{(horizontal slices)}
  \]

- **Volume**
  - Cross Section Method: \( V = \int A(x) \, dx \)
    * Tip: Pick the direction of slicing so that the cross section is easy to find.
    * Tip: If the shape is like a pyramid or a cone, try to let your variable of integration (your \( x \)) be the distance from the vertex of the cone, not from the base. That will make the formulas simpler.
  - Solids of Revolution: Disk and Washer Methods
    * Slice a solid of revolution perpendicularly to the axis of rotation to get disks or washers as the cross-sections.
    * Be careful! If you are not rotating about the \( x \)-axis or the \( y \)-axis, then the radius may be harder to find.
    * To find the radii, label the relevant points on the original graph with their coordinates. Then use those coordinates to compute the radii.
    * The volume of a disk is \( \pi R^2 \Delta x \), where \( R \) is the radius.
    * The volume of a washer is \( (\pi R^2 - \pi r^2) \Delta x \), where \( R \) is the outer radius and \( r \) is the inner radius.
Solids of Revolution: Shell Method

- Rule of thumb: \( y = f(x) \) \( \rightarrow \) Disks, \( x = f(y) \) \( \rightarrow \) Shells

- Slice the solid into thin cylindrical shells \textit{around} the axis of revolution.
- The thickness of the shell will be \( \Delta x \) if the axis is vertical, but \( \Delta y \) if the axis is horizontal.
- The volume of a shell is \((\text{circumference}) \cdot (\text{height}) \cdot (\text{thickness})\) or \((2\pi R)(h)\Delta x\)
- Integrate from the innermost shell to the outermost, or vice versa.

Work

- The key equations: \( \text{work} = \text{force} \cdot \text{distance} \) and \( \text{force} = \text{mass} \cdot \text{acceleration} \)
- The acceleration due to gravity is \( g = 9.8 \text{ m/s}^2 \).
- The force due to a spring is \( kx \) (Hooke’s Law), where \( k \) is the spring constant and \( x \) is the distance the spring has already been stretched from its natural length.
- Slice the work done into pieces, each with the same force.
- Then calculate the work on each piece, and integrate to get the total work.

Chapter 7: Methods of Integration

- Integration By Parts:
  \[
  \int u \, dv = uv - \int v \, du
  \]
  - The goal is to reduce a complicated integral to a (somewhat) simpler integral.
  - You need to split the integrand up into \( u \) and \( v' \); then \( dv = v' \, dx \).
    - \textbf{T}ip: Pick \( u \) so that it gets simpler when differentiated.
    - \textbf{T}ip: Pick \( v' \) so that it doesn’t get much worse when integrated.
    - \textbf{T}ip: If one way doesn’t work, try the other.
    - \textbf{T}ip: Sometimes choosing \( v' = 1 \), i.e. \( dv = dx \), is the right choice!
  - Sometimes you need to do integration by parts two or three times to finish.
  - Sometimes integration by parts, applied twice, returns you to the original integral. (The book calls this “going in a circle.”) Often you can add this to the left-hand side, and then solve for the integral.

Trigonometric Integrals

- Review your trigonometry!
- Recall: \( \frac{d}{dx} \sin x = \cos x \); \( \frac{d}{dx} \cos x = -\sin x \); \( \frac{d}{dx} \tan x = \sec^2 x \); \( \frac{d}{dx} \sec x = \sec x \tan x \)
- Recall: \( \int \sin x \, dx = -\cos x + C \); \( \int \cos x \, dx = \sin x + C \); \( \int \tan x \, dx = \ln | \sec x | + C \)
  - \( \int \sec x \, dx = \ln | \sec x + \tan x | + C \)
• Trig Substitution

– If you see the square root of a quadratic in your integral, such as $\sqrt{x^2 - 7}$ or $\sqrt{3x^2 - 2x + 7}$, you can try trig substitution.
– First check whether ordinary substitution would work more easily. For example, $\int \frac{x}{3x^2 + 19} \, dx$ can be done by setting $u = 3x^2 + 19$.
– Here are the steps to take for a trig substitution. Note that many problems don’t actually require all these steps.

1. If there is a coefficient of $x^2$ in the quadratic, factor it entirely out of the integral.
2. If there is an $x$-term in the quadratic, like $\sqrt{x^2 - 6x + 12}$, then complete the square so it looks something like $\sqrt{(x - 3)^2 + 3}$.
3. Set up your right triangle so that the quadratic arises from the Pythagorean Theorem.
   * Tip: If possible, put $x$ opposite to the angle $\theta$
   * Tip: If possible, put the number adjacent to the angle $\theta$
4. Write $x$, $dx$, and $\sqrt{\cdots}$ in terms of trig functions of $\theta$. Do the substitution to make your integral into a trig integral.
5. Do the trig integral.
6. Finally, translate your answer from $\theta$-stuff back to $x$-stuff. (If your answer includes a $\theta$, you’ll need to use $\sin^{-1}$ or $\cos^{-1}$ or $\tan^{-1}$.)

• Improper Integrals

– If one end of the integral is $\pm \infty$, then we compute it by taking a limit:

\[
\begin{align*}
\int_{-\infty}^{\infty} f(x) \, dx &= \lim_{R \to \infty} \int_{-R}^{R} f(x) \, dx \\
\int_{a}^{\infty} f(x) \, dx &= \lim_{R \to \infty} \int_{a}^{R} f(x) \, dx \\
\int_{-\infty}^{a} f(x) \, dx &= \lim_{R \to -\infty} \int_{R}^{a} f(x) \, dx
\end{align*}
\]

– If that limit exists, we say the integral converges; otherwise we say the integral diverges.
– Recall some useful tools for calculating limits:
   * If the function is continuous, you can just plug in the value using the pseudo-equation $\frac{1}{\infty} = 0$; sometimes you need to prepare the expression first by getting $R$ into denominators.
   * If the numerator and denominator of a fraction are both going to zero or both to $\pm \infty$, then you can use L’Hôpital’s Rule.
– The integral $\int_{1}^{\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1 \\ \infty, & \text{otherwise.} \end{cases}$
Chapter 8: More Applications of the Integral

- **Basic Principle:** For each and every problem in Chapter 8, you set up the integral using the same three steps:
  1. Slice up the thing into thin slices
  2. Find the value (i.e. arc length or surface area or force or moment) of each thin piece
  3. Add up all those values by integrating.

As long as you follow these three steps and use your common sense, you should be able to set up the integral for every problem. Remember that if you are trying to calculate, say, an area $A$, then the total area is $A = \int dA$.

- The **arc length** of a curve is denoted $s$. The following equation is extremely important for this test:
  \[ ds = \sqrt{(dx)^2 + (dy)^2} \]

- To find the **arc length of the function** $y = f(x)$ from $x = a$ to $x = b$, we thus have
  \[ ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2} \, dx = \sqrt{1 + (y')^2} \, dx \]

  Hence the total arc length is
  \[ s = \int ds = \int_a^b \sqrt{1 + f'(x)^2} \, dx \]

  Note that this formula for arc length involves $f(x)$ and $dx$, which makes sense when we are working with the graph of a function $y = f(x)$.

- For problems involving **fluid pressure**, the critical physical laws are
  \[ \text{pressure} = (\text{density}) \cdot (\text{depth}) \quad \text{and} \quad \text{force} = (\text{pressure}) \cdot (\text{area}), \]
  i.e. $p = wh$ and $F = pA$, where $w$ stands for density and $h$ stands for depth. So to calculate total force on a submerged plate, we slice up the plate horizontally (so each piece has constant depth). If the slice at depth $h$ has width $f(h)$, then we have
  \[ dF = p \, dA = (wh)(f(h) \, dh) \]

  so
  \[ F = \int dF = \int_a^b wh \, f(h) \, dh \]
Chapter 11: Parametric and Polar Equations

- Parametric Equations
  - Parametric equations describe a curve by two functions: \( x(t) \) and \( y(t) \), which give the \( x \) and \( y \) coordinates of a particle/bug/pen at time \( t \).
  - We sometimes write just \((x(t), y(t))\) for simplicity’s sake.
  - Convert a parametric curve to its equation in \( x \) and \( y \):
    1. Solve the \( x \)-equation for \( t \)
    2. Plug that into the \( y \)-equation
    3. Simplify
  - To calculate the \textbf{arc length} or \textbf{distance traveled}, we use the fact that
    
    \[
    ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad \text{dt} = \sqrt{x'(t)^2 + y'(t)^2} \, \text{dt}.
    \]
    Then we easily have
    
    \[
    s = \int ds = \int_{a}^{b} \sqrt{x'(t)^2 + y'(t)^2} \, \text{dt}
    \]
    - To calculate the \textbf{speed} of our particle/bug/pen, we know that speed is \( \frac{ds}{dt} \), so we easily have
      \[
      \frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}.
      \]

- Polar Coordinates
  - The point with polar coordinates \((r, \theta)\) is located a distance of \( r \) from the origin, at an angle of \( \theta \) from the positive \( x \)-axis.
  - Be able to convert from polar to rectangular coordinates:
    
    \[
    \begin{align*}
    x &= r \cos \theta \\
    y &= r \sin \theta
    \end{align*}
    \]
  - Be able to convert from rectangular to polar coordinates:
    \[
    \begin{align*}
    r &= \sqrt{x^2 + y^2} \\
    \theta &= \begin{cases} 
    \tan^{-1} \frac{y}{x}, & \text{if } x > 0 \\
    \pi + \tan^{-1} \frac{y}{x}, & \text{if } x < 0
    \end{cases}
    \end{align*}
    \]
  - For heaven’s sake, learn your basic trigonometric values. You should be able to find \( \sin, \cos, \) and \( \tan \) of \( k\pi, \frac{k\pi}{2}, \frac{k\pi}{3}, \frac{k\pi}{4}, \) and \( \frac{k\pi}{6} \), for any integer \( k \). Likewise you should be able to calculate \( \sin^{-1} \) and \( \cos^{-1} \) of the numbers \( \pm \frac{1}{2}, \pm \frac{\sqrt{2}}{2}, \pm \frac{1}{\sqrt{3}}, 0, \) and \( \pm 1 \), and \( \tan^{-1} \) of \( \pm 1, \pm \sqrt{3}, 0, \) and \( \pm \frac{1}{\sqrt{3}} \).
• Polar Equations
  – The graph of a polar equation \( r = f(\theta) \) is drawn by turning through each angle \( \theta \) and plotting the point at a distance of \( f(\theta) \) from the origin.
  – Remember that if \( r \) is negative, we have to go the opposite direction.
  – To calculate the area enclosed by a polar equation \( r = f(\theta) \), note that the area of a little pie-slice of angle \( \theta \) and radius \( r \) is

\[
dA = \text{(fraction of the whole circle)} \cdot \text{(area of whole circle)} = \left( \frac{\pi r^2}{2\pi} \right) \cdot \frac{1}{2} r^2 d\theta;
\]

since \( r = f(\theta) \), the total area is

\[
A = \int dA = \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 d\theta.
\]

**Chapter 10: Convergence of Series**

• Given a series, either in \( \Sigma \)-form or written out, test whether the series converges using one of the following tests.

• **Geometric Series Test:** If the series is \( \sum_{n=0}^{\infty} c r^n \), the convergence depends on \( r \):

\[
\sum_{n=0}^{\infty} c r^n = \begin{cases} \frac{c}{1-r} & \text{if } |r| < 1 \\ \text{DIVERGES} & \text{if } |r| \geq 1. \end{cases}
\]

• **Comparison Test:** If you have two positive series with \( 0 \leq a_n \leq b_n \) eventually, then \( \text{IF } \sum b_n \text{ converges, THEN } \sum a_n \text{ converges too.} \) (But not the other way around!)

  – If you think the series converges, try to increase \( a_n \) to something just a little larger that still converges; then the Comparison Test will prove that \( \sum a_n \) converges too.
  – If you think the series diverges, try to decrease \( a_n \) to something just a little smaller that diverges; then the Comparison Test proves that \( \sum a_n \) diverges too.

• **Limit Comparison Test:** If you think you can see that your positive series \( \sum a_n \) is “basically the same as” some other series \( \sum b_n \) when \( n \) is large, then try computing

\[
\lim_{n \to \infty} \frac{a_n}{b_n}.
\]

\[
\text{If } \lim_{n \to \infty} \frac{a_n}{b_n} > 0, \text{ then the series } \sum a_n \begin{cases} \text{CONVERGES, if } \sum b_n \text{ converges} \\ \text{DIVERGES, if } \sum b_n \text{ diverges} \end{cases}
\]
• **Ratio Test:** This test is especially good if your series $\sum a_n$ has factorials in it. Compute $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$.

| $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$ | Conclusion |
|-----------------|------------------|
| $< 1$ | $\sum a_n$ converges absolutely. |
| $> 1$ | $\sum a_n$ diverges. |
| $= 1$ | this test is useless. |
| doesn’t exist | this test is useless. |

• **Root Test:** This test especially good if your series $\sum a_n$ includes $(something)^n$ in it. Compute $\lim_{n \to \infty} \sqrt[n]{|a_n|}$.

| $\lim_{n \to \infty} \sqrt[n]{|a_n|}$ | Conclusion |
|-----------------|------------------|
| $< 1$ | $\sum a_n$ converges absolutely. |
| $> 1$ | $\sum a_n$ diverges. |
| $= 1$ | this test is useless. |
| doesn’t exist | this test is useless. |

• **Summary of Convergence Tests for this Exam:**

<table>
<thead>
<tr>
<th>Test</th>
<th>Must be positive?</th>
<th>Most useful when...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometric Series</td>
<td>No</td>
<td>Geometric series only</td>
</tr>
<tr>
<td>Comparison Test</td>
<td>Yes</td>
<td>fractions</td>
</tr>
<tr>
<td>Limit Comparison Test</td>
<td>Yes</td>
<td>fractions, rational functions</td>
</tr>
<tr>
<td>Ratio Test</td>
<td>No</td>
<td>factorials or $(\cdots)^n$</td>
</tr>
<tr>
<td>Root Test</td>
<td>No</td>
<td>powers like $(\cdots)^n$</td>
</tr>
</tbody>
</table>

**Chapter 10, cont.: Power Series**

• Be able to find a power series for $f(x)$ about $c$ using the following methods:
  
  – Basic series to know with $c = 0$:

  \[
  \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots
  \]

  \[
  e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots
  \]

  \[
  \sin x = [icky formula] = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots
  \]
– Manipulate power series you know using
  * Substitution
  * Integration
  * Differentiation
  * Multiplication

– If $c \neq 0$ or nothing else will work, use the Taylor Series formula:
  1. Compute the derivatives $f'(x), f''(x), f'''(x)$, etc.;
     extrapolate the general term $f^{(n)}(x)$.
  2. Plug $x = c$ into those derivatives to get $f'(c), f''(c), f'''(c), \ldots$
     and the general term $f^{(n)}(c)$
  3. Now plug those into the formula

\[
    f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots
\]

to get the Taylor series.

• Given a power series $\sum a_n x^n$, find its radius of convergence:

\[
    \text{The radius of convergence of } \sum a_n(x-c)^n \text{ is } \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|
\]

Chapter 9: Differential Equations

• Be able to identify the order of a differential equation. This is the highest derivative of $y$ that appears in the equation.

• If a differential equation is first order, you might be able to solve it by separating the variables. This should be the first thing you try.

  1. First, write $y'$ as $\frac{dy}{dx}$.
  2. Then manipulate the equation until all the $y$'s (including the $dy$) are on one side of the equation, and all the $x$'s (including the $dx$) are on the other side.
  3. Make sure the $dx$ and $dy$ are not in the denominator! (If they are, take the reciprocal of both sides to fix it.)
  4. Now integrate both sides. You only need to put a “$+C$” on one side of the resulting equation.
  5. Finally, solve for $y$. 
• If you can’t separate the variables, another approach is to use an integrating factor.

1. Manipulate the equation until it looks like

\[ y' + (\text{something without any } y\text{'s}) \cdot y = (\text{something without any } y\text{'s}), \]

or in other words \( y' + A(x)y = B(x). \)

2. Use the formula \( e^{\int A(x) \, dx} \) to get the integrating factor \( \alpha(x). \) (If that formula gives you absolute value signs, try ignoring them—it usually works.)

3. Multiply both sides of the differential equation by your integrating factor \( \alpha(x). \)

4. Double-check that the left-hand side is the derivative of \( \alpha(x) \cdot y. \)

5. Now integrate both sides of the equation.

6. Solve for \( y. \)

• Solve initial value problems:

1. Solve the differential equation (using one of the two methods above). Your answer will have a constant \( C \) in it somewhere.

2. The initial value condition says something like \( y(3) = 7. \) Plug \( x = 3 \) and \( y = 7 \) into the equation.

3. Now solve for \( C. \)

4. Your final answer is the solution to the differential equation, using that particular value for \( C. \)