Subtraction, Summary, and Subspaces

Linear Algebra, Fall 2008

1 Subtraction

The vector space axioms talk about only two operations: addition and scalar multiplication. Nevertheless, the idea of subtracting two vectors is hidden inside those axioms. Among the real numbers, if you wanted to rewrite the subtraction $7-3$ in terms of addition, you would write $7+(-3)$. An analogous idea works perfectly well in a vector space.

Definition 1.1. *Let V be a vector space and let $\mathbf{u}, \mathbf{v} \in V$. Then we define the difference $\mathbf{u}-\mathbf{v}$ to be the sum $\mathbf{u}+(-\mathbf{v})$.*

This is really just a notational convenience, a way to save time and ink. Linear algebra would get awfully annoying if we had to keep writing $\mathbf{u}+(-\mathbf{v})$ all the time, so we write it as a subtraction $\mathbf{u}-\mathbf{v}$ instead. Fortunately, subtraction of vectors has most of the properties you are used to in subtraction of real numbers:

Theorem 1.2. *Let V be a vector space and let $\mathbf{v}, \mathbf{w}, \mathbf{x} \in V$. Let r and s be any scalars. Then the following are all true:*

(a) $(\mathbf{v}-\mathbf{w})+\mathbf{w}=\mathbf{v}$
(b) $\mathbf{v}-\mathbf{0}=\mathbf{v}$
(c) $\mathbf{0}-\mathbf{v}=-\mathbf{v}$
(d) $\mathbf{v}-\mathbf{w}=-\mathbf{w}+\mathbf{v}$
(e) $-\mathbf{v}-\mathbf{w}=-\mathbf{w}-\mathbf{v}$
(f) $-(\mathbf{v}+\mathbf{w})=-\mathbf{v}-\mathbf{w}$
(g) $\mathbf{v}-(-\mathbf{w})=\mathbf{v}+\mathbf{w}$
(h) $(\mathbf{v}+\mathbf{w})-\mathbf{w}=\mathbf{v}$
(i) $\mathbf{v}-(\mathbf{w}+\mathbf{x})=(\mathbf{v}-\mathbf{w})-\mathbf{x}$
(j) $\mathbf{v}-(\mathbf{w}-\mathbf{x})=(\mathbf{v}-\mathbf{w})+\mathbf{x}$
(k) $\mathbf{v}-(-\mathbf{w}+\mathbf{x})=(\mathbf{v}+\mathbf{w})-\mathbf{x}$
(l) $\mathbf{v}-(-\mathbf{w}-\mathbf{x})=(\mathbf{v}+\mathbf{w})+\mathbf{x}$
(m) $r(\mathbf{v}-\mathbf{w})=r\mathbf{v}-(r\mathbf{w})$
(n) $(r-s)\mathbf{v}=r\mathbf{v}-(s\mathbf{v})$

Each of the parts of Theorem 1.2 is very easy to prove, and some are listed as exercises.

2 Summary of Arithmetic

At this point, if we've read all the examples and done all the exercises in the *Sets, Logic, and Proof* handout, the *Introduction to Vector Spaces* handout, and the preceding section, we have just about justified the following claim:

> Arithmetic in a vector space follows almost all the rules we expect from our experience with real-number arithmetic.

From here on out, therefore, we will stop worrying about parentheses and negatives. We've paid our dues, we've done the tedious work of justifying basic arithmetic from the axioms, and we can reap the rewards. We'll just write $(\mathbf{v}+\mathbf{w})+\mathbf{x}$ as $\mathbf{v}+\mathbf{w}+\mathbf{x}$ and $-\mathbf{x}+\mathbf{v}$ as $\mathbf{v}-\mathbf{x}$; we'll skip steps and jump directly from $\mathbf{v}+\mathbf{w}=\mathbf{x}+\mathbf{w}$ to $\mathbf{v}=\mathbf{x}$; we won't bother distinguishing among $(-r)\mathbf{v}$, $-(r\mathbf{v})$, and $-r\mathbf{v}$.

3 Subspaces

Consider the vector space $\mathbb{R}^3$ of ordered triples of real numbers, and let

$$U = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}.$$

If we check the ten axioms, we will see that U is itself a vector space! That is, the vector space $\mathbb{R}^3$ has other vector spaces living inside of it. We'll refer to this situation often, so let's give it a name.

Definition 3.1. *Let V be a vector space and let W be a subset of V. If W, using the same addition and scalar multiplication as are used in V, is itself a vector space, then we say W is a* **subspace** *of V.*

Not all subsets of $\mathbb{R}^3$ are subspaces, though. Let

$$T = \left\{ \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} : x, y \in \mathbb{R} \right\}.$$

Then if you check the axioms, you will quickly see that T is not a vector space. Not only is T closed under neither addition (since $(2,3,1) + (4,6,1) \notin T$) nor scalar multiplication (since $\pi(2,3,1) \notin W$), but T doesn't contain inverses or even a zero vector! On the other hand, T does satisfy the other six axioms, and in fact it doesn't really have a choice about them. Look at the following chart.

Vector Space Axioms

2. For all $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. 3. For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$. 7. For all $\mathbf{u}, \mathbf{v} \in V$ and for any scalar c, $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$. 8. For any $\mathbf{u} \in V$ and for all scalars c and d, $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$. 9. For any $\mathbf{u} \in V$ and for all scalars c and d, $c(d\mathbf{u}) = (cd)\mathbf{u}$. 10. For any $\mathbf{v} \in V$, $1\mathbf{u} = \mathbf{u}$.	1. For all $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} \in V$ 4. There exists an element $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$. 5. For each $\mathbf{u} \in V$, there is an element $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. 6. For all $\mathbf{u} \in V$ and for any scalar c, $c\mathbf{u} \in V$.

The axioms in the left-hand column of the chart state that *all* vectors in V satisfy certain equalities. If W is a subset of V, then all vectors in W are also a vectors in V, so they automatically satisfy Axioms 2, 3, 7, 8, 9, and 10. Therefore to check whether W is a subset of V, we only need to check Axioms 1, 4, 5, and 6 with "W" in place of "V". Moreover, if Axiom 6 is satisfied, then for any $w \in W$, it follows that $-w = (-1)w \in W$, so Axiom 5 is automatically satisfied. Finally, note if W has a zero vector, it must be the zero vector of V too. (See Exercise 4.) In short, to check whether a sub*set* of V is a sub*space*, we only need to check three things:

Lemma 3.2. *Let V be a vector space and let W be a subset of V. If the following three conditions are satisfied, then W is a subspace of V.*

1. $\mathbf{0} \in W$.
2. *W is closed under addition (i.e., $\mathbf{w} + \mathbf{x} \in W$ for any $\mathbf{w}, \mathbf{x} \in W$).*
3. *W is closed under scalar multiplication (i.e., $c\mathbf{w} \in W$ for any $\mathbf{w} \in W$ and any scalar c).*

Lemma 3.2 makes it very easy to check whether a subset is a subspace, as the following example illustrates.

Example 1. *We know that the set $\mathcal{P}$ of all polynomials is a vector space. Let $\mathcal{P}_2$ be the set of polynomials of degree 2 or lower (with real coefficients), and show that $\mathcal{P}_2$ is a subspace of $\mathcal{P}$.*

Proof. First, note that $\mathcal{P}_2 = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$, which is a subset of $\mathcal{P}$. Then

- The zero polynomial $0 = 0x^2 + 0x + 0 \in \mathcal{P}_2$.
- Choose any two elements of $\mathcal{P}_2$, say $ax^2 + bx + c$ and $dx^2 + ex + f$. Then

  $$(ax^2 + bx + c) + (dx^2 + ex + f) = (a + d)x^2 + (b + e)x + (c + f) \in \mathcal{P}_2.$$

  Hence $\mathcal{P}_2$ is closed under addition.
- Let r be any scalar and consider any element $ax^2 + bx + c$ of $\mathcal{P}_2$. Then

  $$r(ax^2 + bx + c) = (ra)x^2 + (rb)x + (rc) \in \mathcal{P}_2,$$

  so $\mathcal{P}_2$ is closed under scalar multiplication.

Therefore $\mathcal{P}_2$ is a subspace of $\mathcal{P}$. □

Exercises

1. Prove Theorem 1.2. Let V be a vector space, let $\mathbf{v}, \mathbf{w}, \mathbf{x} \in V$, and let r and s be scalars. Prove that

   (a) $(\mathbf{v} - \mathbf{w}) + \mathbf{w} = \mathbf{v}$

   (b) $\mathbf{v} - \mathbf{0} = \mathbf{v}$

   (c) $\mathbf{0} - \mathbf{v} = -\mathbf{v}$

   (d) $\mathbf{v} - \mathbf{w} = -\mathbf{w} + \mathbf{v}$

   (e) $-\mathbf{v} - \mathbf{w} = -\mathbf{w} - \mathbf{v}$

   (f) $-(\mathbf{v} + \mathbf{w}) = -\mathbf{v} - \mathbf{w}$

   (g) $\mathbf{v} - (-\mathbf{w}) = \mathbf{v} + \mathbf{w}$

   (h) $(\mathbf{v} + \mathbf{w}) - \mathbf{w} = \mathbf{v}$

   (i) $\mathbf{v} - (\mathbf{w} + \mathbf{x}) = (\mathbf{v} - \mathbf{w}) - \mathbf{x}$

   (j) $\mathbf{v} - (\mathbf{w} - \mathbf{x}) = (\mathbf{v} - \mathbf{w}) + \mathbf{x}$

   (k) $\mathbf{v} - (-\mathbf{w} + \mathbf{x}) = (\mathbf{v} + \mathbf{w}) - \mathbf{x}$

   (l) $\mathbf{v} - (-\mathbf{w} - \mathbf{x}) = (\mathbf{v} + \mathbf{w}) + \mathbf{x}$

   (m) $r(\mathbf{v} - \mathbf{w}) = r\mathbf{v} - (r\mathbf{w})$

   (n) $(r - s)\mathbf{v} = r\mathbf{v} - (s\mathbf{v})$

2. Let V be a vector space and let $\mathbf{v}, \mathbf{w}, \mathbf{x} \in V$. Prove that if $\mathbf{v} + \mathbf{x} = \mathbf{w} + \mathbf{x}$, then $\mathbf{v} = \mathbf{w}$.

3. Let V be a vector space, let $\mathbf{v}, \mathbf{w} \in V$, and let r and s be scalars. Suppose $\mathbf{v} \neq \mathbf{0}$. Prove that if $r\mathbf{v} + \mathbf{w} = s\mathbf{v} + \mathbf{w}$, then $r = s$.

4. Let V be a vector space, whose additive identity is denoted $\mathbf{0}$. Let W be a subset of V, and suppose W has a "zero vector" $\mathbf{z}$ such that

$$\mathbf{w} + \mathbf{z} = \mathbf{w} \text{ for all } \mathbf{w} \in W.$$

Prove that $\mathbf{z} = \mathbf{0}$.

5. Is $\{(x, y, z) : x + y + z = 0\}$ a subspace of $\mathbb{R}^3$? Justify your answer.

6. Is $\{(x, y) : x + y = 1\}$ a subspace of $\mathbb{R}^2$? Justify your answer.

7. (a) What is the smallest subspace of $\mathbb{R}^3$ containing the vector $(1, 0, 0)$?
   (b) Let V be a vector space and let $\mathbf{v} \in V$. What is the smallest subspace of V containing $\mathbf{v}$?

8. Find a subset of $\mathcal{P}$ that is closed under scalar multiplication but not closed under addition.

9. Is $\{f(x) \in \mathcal{P} : f(2008) = 0\}$ a subspace of $\mathcal{P}$? Justify your answer.

10. Let V be a vector space, suppose W is a subspace of V, and suppose T is a subspace of W. Prove that T is a subspace of V.

11. Let V be a vector space and let $\mathbf{v}, \mathbf{w} \in V$. Show that $\{a\mathbf{v} + b\mathbf{w} : a, b \in \mathbb{R}\}$ is a subspace of V.